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Dynamic Hypermedial Aids (DHA) in Teaching Notable Products

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Abstract: This study proposes a novel pedagogical framework based on Dynamic Hypermedial Aids (DHA), integrating socioconstructivist principles, the Van Hiele model of geometric reasoning, and information and communication technologies (ICT) to enhance the teaching of notable algebraic products. The main contribution of the study lies in the development of a structured DHA-based instructional model that extends beyond conventional ICT-supported approaches by combining geometric visualization, variational thinking, and adaptive learning sequences.

The research adopts a qualitative analytical–interpretive design involving a purposive sample of 32 eighth-grade students. Data were collected through participant observation, video recordings, and activity logs, and analyzed using thematic coding and methodological triangulation.

The findings indicate improvements in students' conceptual understanding of algebraic–geometric relationships, particularly in interpreting binomial squares through geometric representations. Additionally, increased levels of student engagement and the emergence of more autonomous learning behaviors were observed.

Overall, the study highlights the potential of DHA as an effective pedagogical tool for fostering deeper mathematical understanding and supporting active learning processes in algebra education.

Keywords: Dynamic Hypermedial Aids (DHA); Mathematics Education; Algebraic Thinking; Geometric Reasoning; Variational Thinking; ICT in Education; Instructional Design; Secondary Education.

动态超媒体辅助（DHA）在重要乘积教学中的应用

摘要：本研究提出了一种基于动态超媒体辅助（Dynamic Hypermedial Aids, DHA）的创新教学框架，将社会建构主义原则、Van Hiele几何推理模型以及信息与通信技术（ICT）相结合，以提升重要代数公式的教学效果。该研究的主要贡献在于构建了一种结构化的DHA教学模型，该模型超越了传统以信息技术支持为主的教学方式，通过整合几何可视化、变式思维以及自适应学习序列，优化学生的学习过程。

本研究采用定性分析—解释性研究方法，选取32名八年级学生作为目的性样本。数据通过参与式观察、视频记录及学习活动日志收集，并运用主题编码与方法三角验证进行分析。

研究结果表明，学生在代数—几何关系的概念理解方面显著提升，尤其是在通过几何表征理解二项式平方方面表现出更高水平。同时，学生的学习参与度明显提高，并呈现出更强的自主学习行为。

总体而言，本研究表明，动态超媒体辅助（DHA）作为一种教学工具，在促进数学深度理解和支持主动学习方面具有重要潜力，特别是在代数教学领域。

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关键词：动态超媒体辅助（DHA）；数学教育；代数思维；几何推理；变式思维；信息与通信技术；教学设计；中等教育

I. Introduction

During the teaching and learning processes, students develop mathematical skills and competencies both inside and outside the classroom. When these competencies are acquired outside the classroom, often based on prior knowledge and experiences closely linked to the development of mathematical thinking constructed autonomously in real-life contexts, the traditional object of teaching may be reconsidered and adapted through a systematized pedagogical model tailored to a specific educational community. This is particularly relevant given that students' understandings and perceptions of mathematical activities can vary significantly. Therefore, it is essential for teachers to recognize and incorporate the concepts that students associate with their everyday experiences.

In this context, teachers play a crucial role in creating meaningful learning environments that connect formal mathematical concepts with those encountered in other contexts. From this perspective, school learning should be understood as an active process from the student's point of view, in which learners construct, modify, enrich, and diversify their knowledge structures based on the meaning they attribute both to the content and to the learning process itself [1].

Teachers are therefore required to transform their instructional practices by incorporating innovative activities that enhance student engagement in the teaching and learning process. In this regard, it is pertinent to include the use of ICT tools within pedagogical design. According to the Basic Learning Rights (DBA), Version 2, students are expected to propose, compare, and apply inductive procedures, as well as use algebraic language to formulate and test conjectures across different contexts [2]. In this sense, students represent numerical relationships through algebraic expressions.

Accordingly, teachers can integrate ICT tools supported by mathematics education theories through the development of didactic sequences focused on notable products. Within this framework, the present study seeks to address the following research question: What are the didactic contributions of Dynamic Hypermedial Aids (DHA) in teaching notable products and their geometric applications to eighth-grade students, particularly in terms of the use and creation of educational materials, the adaptation of resources to contextual needs, and the implementation of effective communicative strategies in the classroom? To guide this research, one general objective and three specific objectives are proposed.

Despite the growing integration of ICT in mathematics education, there remains a lack of structured pedagogical models that effectively combine technological, pedagogical, and mathematical knowledge. This study aims to address this gap by proposing a DHA-based pedagogical design that explicitly links digital tools with geometric reasoning and algebraic thinking.

II. State of the art

The following studies are considered relevant to the present research, as they are situated within the field of mathematics education and address problems in school contexts comparable to those examined in this work.

The article “Teaching and Learning in Problem Solving: Notable Products of Reflection” by Alfonso Jiménez, Laura Parra, and Hollman Camacho, derived from the research project “Mathematical Practices in Problem Solving on Factorization,” aims to identify the characteristics of teaching and learning practices related to factorization and notable products. The study analyzes three school textbooks (Ortiz, 2009; Bautista, 2003; Baldor, 1986) and includes a questionnaire consisting of 11 open- and closed-ended questions administered to 191 eighth-grade students from the Institución Educativa Técnica Antonio Nariño (Moniquirá, Boyacá). The research adopts a mixed-methods approach, incorporating an onto-semiotic perspective and content analysis.

The findings indicate a high level of student interest in knowledge acquisition, as well as positive perceptions regarding the use of technology in algebra classes. However, the results also reveal significant limitations in traditional algebra teaching models. These models tend to prioritize lecture-based instruction, with minimal integration of ICT, a predominance of individual activities lacking interactive or engaging elements, and low student motivation. Students report difficulties in understanding the content, largely due to an overreliance on teacher-centered explanations, limited use of geometric representations, and insufficient incorporation of didactic resources that encourage active participation.

From this perspective, the present study considers this article particularly relevant, as it demonstrates that instruction based solely on traditional tools (e.g., pencil and paper) may be insufficient for effective algebra learning. Such approaches often disconnect factorization from other mathematical concepts and restrict the use of multiple representations that could support deeper understanding. The study also highlights the importance of integrating ICT tools, such as GeoGebra, which can enhance the teaching of notable products by providing dynamic visualizations, Cartesian representations, and opportunities for the development of spatial and variational thinking. These aspects are central to the pedagogical design proposed in this research, which incorporates problem-solving activities supported by geometric representations of algebraic expressions using GeoGebra.

In addition, the Master’s thesis by Sandra Isabel Salazar Giraldo (2017), entitled “Dynamic Hypermedial Aids (DHA) for Teaching Algebraic Expressions in the Introduction to Algebra with Eighth-Grade Students at the Institución Educativa San Pablo (Pueblo Rico, Risaralda),” provides an important reference for this study. This work presents an innovative didactic proposal in which ICT is used as a mediating tool in the teaching and learning process.

The thesis emphasizes the design of didactic sequences that promote collaborative learning within a socio-constructivist framework, as well as the development of teaching strategies grounded in autonomous learning and problem-based learning. The results demonstrate that the implementation of DHA strengthens teacher-student interaction and fosters a participatory learning environment. Furthermore, the use of multiple representations, including illustrations, examples, and demonstrations, supports the understanding of algebraic expressions and contributes to more meaningful learning experiences.

III. Theoretical framework

A. Socioconstructivism in Mathematics Teaching

Vygotsky argues that every mental mathematical operation originates as an interpersonal activity. This principle, known as the general genetic law of cultural development, posits that all higher cognitive functions emerge on two levels: first on the interpsychological plane (between individuals) and subsequently on the intrapsychological plane (within the individual). This framework is equally applicable to mathematics learning, where children’s initial mathematical knowledge develops through social interaction, particularly through activities such as counting objects.

In this process, interaction between the child and a more knowledgeable individual, typically an adult, is essential. Arithmetic operations initially take place through the manipulation of tangible objects; however, with continuous guidance, these operations gradually become internalized, with symbols replacing concrete objects. In this sense, the development of mathematical thinking is a socially mediated process that requires guided participation.

Thus, the construction of mathematical knowledge is closely linked to the involvement of a more experienced individual, whose support contributes to the learner’s zone of proximal development. Cognitive processes are therefore mediated through interaction with others who possess greater expertise in mathematical representations and language, facilitating the internalization and reflection of mathematical activity, as noted by Gómez [3].

B. Dynamic Hypermedial Aids (DHA)

Amador Montaño & others [4] propose that: "A Dynamic Hypermedial Aid is a multimedia product, provided with a hypertextuality system with an open structure, that develops specific content with pedagogical strategies based on socioconstructivism"; that is, a pedagogical proposal in ICT that allows knowing the pedagogical intentions of both the teacher and the student, in dynamic processes in the classroom.

C. Learning Theories

1) Autonomous Learning:

This process involves students establishing their own learning rhythms and styles, as well as identifying the elements they consider necessary for the development of their learning. In this context, learners assume an active and responsible role in constructing their knowledge and developing their skills. They also engage in strategies that enable them to regulate information processing related to both cognitive and socio-affective learning processes [5].

2) Autonomous Learning in Mathematics Teaching

Students' autonomous work is considered effective when it is accompanied by the harmonious development of complementary competencies, as well as habits of independent and self-directed learning. In this context, educational institutions promote classroom practices aimed at strengthening students' abilities in information searching, ICT integration, problem-solving, analysis, synthesis, reasoning, and argumentation [6].

In line with this perspective, the teaching and learning of mathematics should be coherently aligned with the design and implementation of learning activities intended to achieve specific educational objectives. Within this framework, assessment is understood as an integral component of autonomous learning, enabling the identification and development of cognitive, metacognitive, and motivational strategies.

Accordingly, the assessment instruments proposed in this study, supported by structured instructional scaffolding, facilitate the monitoring of both teaching practices and student learning processes. These instruments contribute to the development of learners' autonomy by supporting the acquisition of skills necessary for effective participation in the teaching and learning process.

3) Collaborative Learning:

According to Cabrera [7], "collaborative learning is defined as a situation in which a group of individuals establishes a shared commitment to accomplish a task, where coordination and interaction among participants are essential for achieving a common goal."

This definition highlights an approach that promotes the alignment of ideas, principles, and practices to achieve collective objectives, supporting the social construction of knowledge. At the same time, it emphasizes individual responsibility within the group, enabling each member to contribute actively, seek support when needed, and critically evaluate both their own performance and that of the group.

4) Collaborative Learning in Mathematics Teaching

In the context of mathematics education, effective collaborative learning requires teachers to carefully structure and sequence learning activities, prioritizing them according to the group's context and shared objectives. In doing so, teachers facilitate access to mathematical knowledge and create conditions that support the development of new learning based on students' collaborative skills and prior knowledge [8].

Furthermore, it is essential for teachers to provide clear guidance throughout all stages of the learning process, including the beginning, development, and conclusion, in order to foster student motivation and active participation. Such clarity encourages students to share ideas, discuss interpretations, and engage in meaningful exchanges grounded in their prior knowledge. Consequently, each group member is recognized as a valuable contributor to the successful completion of collaborative tasks [9].

5) Problem-Based Learning

Barrows [10] defines problem-based learning (PBL) as "a learning method based on the use of problems as a starting point for the acquisition and integration of new knowledge." Within this approach, traditional lecture-based instruction is replaced by student-centered learning, in which the teacher assumes the role of a facilitator and students take responsibility for their own learning processes.

In this context, students are encouraged to engage actively in solving mathematical problems, which requires the design of challenging tasks that stimulate curiosity and inquiry. Therefore, it is essential for teachers to present meaningful problem situations and provide appropriate pedagogical support to ensure effective collaborative work. This approach promotes the development of conscious and reflective learning processes, fostering deeper understanding and active participation.

6) Problem-Based Learning in Mathematics Teaching

For the application of a problem-based learning methodology, it is important to mention some aspects that should be considered in a method for the problem-solving work process, as expressed by Moust, Bouhuijs, and Schmidt [11]:

- Clarify concepts and terms: It aims to clarify possible terms in the problem text that are difficult (technical) or vague, so that the entire group shares their meaning.
- Define the problem: It is a first attempt to identify the problem that the text poses. Subsequently, after steps 3 and 4, one can return to this first definition if considered necessary.

- Analyze the problem: In this phase, students contribute all the knowledge they possess about the problem as formulated, as well as possible connections that could be plausible. The emphasis in this phase is more on the quantity of ideas than on their veracity (brainstorming).
- Make a systematic summary with several explanations for the analysis of the previous step: Once the greatest number of ideas about the problem has been generated, the group tries to systematize and organize them, highlighting the relationships that exist between them.
- Formulate learning objectives: At this moment, students decide which aspects of the problem require further investigation and better understanding, which will constitute the learning objectives that will guide the next phase.
- Synthesis of the collected information and preparation of the report on the acquired knowledge: The information provided by the different members of the group is discussed, contrasted, and finally, pertinent conclusions are drawn for the problem.

D. Van Hiele Model

This model allows developing geometric reasoning through guidelines to organize the educational curriculum, explains the evolution of learning through five consecutive levels, and establishes the phases that the student must fulfill to advance from one level to another. Therefore, Fouz [12] expresses that:

When moving up a level, the knowledge that was implicit in the previous level becomes explicit in the student, which indicates that the degree of understanding and mastery of knowledge increases in this way. This makes the objects of work at this higher level extensions of those at the previous level.

The model therefore illustrates how teachers can facilitate and guide the learning process with the aim of improving students' geometric reasoning. Progression from one level to another depends largely on the quality and sequencing of prior instruction. For this reason, the organization of content and pedagogical materials is essential to support the systematic development of the levels proposed by Van Hiele.

In this framework, mathematical concepts are not acquired at a single level; rather, each level is characterized by its own language, symbols, and systems of relationships. Consequently, the language used by students should correspond to their current level of reasoning, and instructional materials must be carefully designed to align with the phases of the learning process.

Now, the authors Fouz and De Donosti (2005), Jaime (1993), Jaime and Gutiérrez (1994), and Beltranetti, Esquivel, and Ferrari (2005), cited in Vargas & Gamboa [13], describe the Van Hiele model recognizing five levels of geometric reasoning:

- Level 1. Recognition or visualization
- Level 2. Analysis
- Level 3. Informal deduction or order
- Level 4. Deduction
- Level 5. Rigor

1) Van Hiele Learning Phases

The model proposes five learning phases that guide teachers in designing and organizing instruction aimed at facilitating students' progression through levels of geometric reasoning. These phases are: Information, Directed Orientation, Explicitation, Free Orientation, and Integration.

These phases enable teachers to assess students' reasoning processes through observation and analysis of the arguments used to justify solutions in mathematical activities. In this way, they provide a structured framework for identifying students' current level of understanding and supporting their advancement to higher levels of reasoning.

Accordingly, teachers are required to employ appropriate tools, methodologies, and theoretical frameworks that effectively support the instructional process and promote meaningful learning among students.

E. TPACK Model

The Technological Pedagogical Content Knowledge (TPACK) model conceptualizes the types of knowledge that teachers need to effectively integrate information and communication technologies (ICT) into classroom practice. It encompasses three core domains of knowledge: technological (TK), pedagogical (PK), and content (CK), which collectively support the development of effective teaching and learning processes.

The implementation of the TPACK model in educational contexts is particularly important, as it integrates knowledge of students and their learning environments, enabling teachers to design flexible and context-responsive pedagogical strategies. This approach facilitates students' understanding by helping them construct meaningful connections with the content they are learning.

Originally proposed by Mishra and Koehler, the TPACK framework is composed not only of the three primary knowledge domains—Content Knowledge (CK), Pedagogical Knowledge (PK), and Technological Knowledge (TK)—but also of their intersections: Pedagogical Content Knowledge (PCK), Technological Content Knowledge (TCK), and Technological Pedagogical Knowledge (TPK). These interconnected domains form a comprehensive framework for analyzing and guiding the integration of technology in teaching practice.

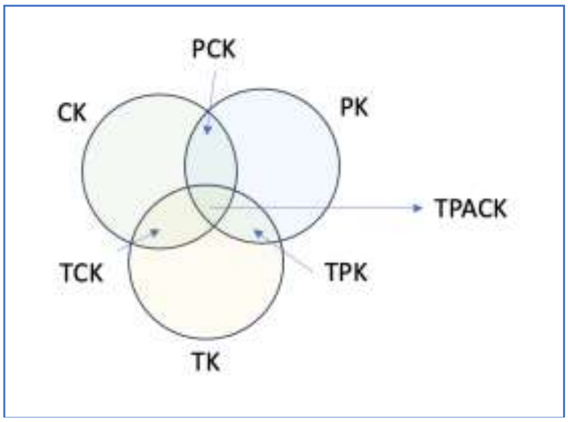

Figure 1. TPACK Model (Own source)

F. The Knowledge Quartet (KQ) – Tim Roland

The Knowledge Quartet (KQ) model, developed at the University of Cambridge in 2002, focuses on teachers' classroom practices in relation to their mathematical knowledge and pedagogical content knowledge. It emphasizes how teachers use subject-specific knowledge and instructional strategies to support effective teaching and learning processes.

Within this framework, the model highlights the importance of designing and implementing innovative didactic sequences, particularly those based on problem situations. It also underscores the need for teachers to employ a range of pedagogical strategies and appropriate resources that facilitate students' ability to connect prior knowledge with new learning experiences.

To effectively promote mathematics teaching and learning, the KQ model identifies four key dimensions [14]:

- Foundation
- Transformation
- Connection
- Contingency

G. Mathematical Knowledge for Teaching (MKT)

The model called MKT (Mathematical Knowledge for Teaching), developed by the research group coordinated by Deborah Ball at the University of Michigan, is defined as "the mathematical knowledge that the teacher uses in the classroom to produce instruction and growth in the student" [15]. That is, according to this model, the teacher must possess content knowledge to exercise mathematics teaching and is classified as follows:

- Common Content Knowledge
- Specialized Content Knowledge
- Knowledge of Content and Students
- Knowledge of Content and Teaching

H. Geometric Relationships and Algebraic Language

1) Geometric Relationships:

According to Bishop [16], Geometry models the space we perceive. That is, many mathematical relationships and concepts are found in the environment; therefore, their study is immersed in the forms observed in daily life to later define them according to their properties as geometric figures [17].

2) Algebraic Notation:

The symbols used in Algebra to represent quantities are numbers and letters.

3) Algebra Signs:

The signs used in Algebra are of three classes: Operation signs, Relation signs, and Grouping signs.

- Operation signs: In Algebra, the same operations as in arithmetic are verified with quantities: Addition, Subtraction, Multiplication, Division, Raising to Powers, and Root Extraction.
- Relation signs: These signs are used to indicate the relationship that exists between two quantities.
- Grouping signs: They are parentheses (), square brackets [], braces {}, and the vinculum.

4) Coefficient:

In the product of two factors, either factor is called the coefficient of the other factor.

5) Algebraic Expression:

It is the representation of an algebraic symbol or one or more algebraic operations.

- Monomial: It is an algebraic expression consisting of a single term.
- Polynomial: It is an algebraic expression consisting of more than one term. If it has two terms, it is called a binomial, and if it has three terms, it is called a trinomial.
- Term: It is an algebraic expression consisting of a single symbol or several symbols not separated from each other by the sign + or -. The elements of a term are four: the sign, the coefficient, the variable (letter), and the exponent.

6) Sign:

By sign, positive terms are those preceded by the + sign, and negative terms are those preceded by the - sign.

7) Coefficient:

It is the first of the factors of the term and is numerical.

8) Literal Part:

It is constituted by the letters (variables) present in the term.

9) Degree:

The degree of a term concerning a variable (letter) is the exponent of that letter.

I. Properties of Quadrilaterals and Like Terms

1) Quadrilateral:

It is a polygon with four sides. In it, pairs of opposite sides (that have no points in common) and pairs of consecutive sides (that have a point in common, the vertex) are identified.

2) Square:

All its sides are congruent, and all its angles have the same measure.

3) Rectangle:

All its angles are right angles.

4) Property of Parallelograms:

Opposite sides of a parallelogram are congruent.

5) Like Terms:

Two or more terms are similar when they have the same literal part, that is, when they have the same letters affected by the same exponents.

J. Multiplication of Algebraic Expressions, Area of Square and Rectangle

1) Definition:

It is an operation whose purpose, given two quantities called multiplicand and multiplier, is to find a third quantity called product, which is, with respect to the multiplicand, in absolute value and sign, what the multiplier is concerning the positive unit. The multiplicand and the multiplier are called factors of the product.

2) Area of Plane Figures (Square and Rectangle):

To determine the area of rectangles, their dimensions must be multiplied; that is, base times height.

K. Notable Products

1) Definition:

It is a multiplication whose result can be identified immediately without needing to perform the complete procedure. Notable products are very useful for streamlining the development of algebraic operations [18]. "Notable products are certain products that follow fixed rules and whose result can be written by simple inspection, that is, without verifying the multiplication" [19].

2) Square of a Binomial:

Square of a Binomial (Addition)

The square of the sum of two terms equals the square of the first term plus twice the product of the first term by the second, plus the square of the second term.

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Square of a Binomial (Subtraction)

The square of the difference of two terms equals the square of the first term minus twice the product of the first term by the second, plus the square of the second term.

$$(a-b)^2 = a^2 - 2ab + b^2.$$

3) Product of Sum by Difference:

The product between the sum of two terms and their difference equals the square of the first term minus the square of the second.

$$(a+b)(a-b) = a^2 - b^2.$$

4) Product of Binomials with a Common Term:

The product of two binomials with a common term equals the square of the common term plus the product of the sum of the non-common terms and the common term, plus the product of the non-common terms.

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

IV. Methodology

The This study adopts a qualitative, analytical–interpretive research design to examine the objectives and outcomes associated with the following research question: What are the didactic contributions of Dynamic Hypermedial Aids (DHA) in teaching notable products and their applications to eighth-grade students, particularly in terms of the use and creation of educational materials, the adaptation of resources to contextual needs, and the implementation of communicative strategies in the classroom?

Within this framework, the study analyzes teachers' mathematical and pedagogical knowledge as manifested in the implementation of a didactic sequence. It also explores the pedagogical contributions of integrating DHA and ICT into the teaching and learning process, with particular attention to their role in the design, production, and contextual adaptation of educational materials.

Participants

The study involved 32 eighth-grade students (ages 12–14) from a public school in Pereira, Colombia. Participants were selected through intentional sampling based on their enrollment in the target course.

Data Collection

Data were collected through three main instruments:

Participant observation logs

Video recordings of classroom sessions

Student activity artifacts

Data Analysis

The data analysis followed a three process:
Open coding of observational and video data
Categorization based on learning theories (autonomous, collaborative, PBL)
Triangulation across data sources to ensure validity
Interpretation criteria were based on evidence of conceptual understanding, student interaction, and problem-solving performance.

A. Research Context

The research will be conducted with eighth-grade students of the Institución Educativa Augusto Zuluaga in the municipality of Pereira, Risaralda, single campus. It has two shifts; but in the morning, secondary education is present for the academic process based on an environmental pedagogical approach.

B. Research Techniques and Instruments

- Activity Log
- Video Recordings
- Participant Observation

C. Pedagogical Design

Teaching and learning processes require a high degree of commitment to identify the problems affecting the good development of the content acquisition process; but also evidence that new practices need a more significant approach; therefore, the pedagogical design matrix for teaching the application of notable products is presented based on the socio-constructivist approach and learning theories with the incorporation of the three levels of Van Hiele reasoning and the traceability of variational thinking; thus, the classification with number 5 indicates the relationship considered most relevant to meet the learning objectives:

Teaching and learning processes require a high degree of commitment to identify the problems affecting the good development of the content acquisition process; but also evidence that new practices need a more significant approach; therefore, the pedagogical design for teaching the application of notable products is based on the socio-constructivist approach and learning theories with the incorporation of the three levels of Van Hiele reasoning and the traceability of variational thinking.

The design matrix establishes the relationships between learning theories (Autonomous Learning, Collaborative Learning, Problem-Based Learning) and the key components of socio-constructivism (Scaffolding, Adjusted Help, Zone of Proximal Development, Co-construction of Shared Meanings) as well as the first two levels of Van Hiele's geometric reasoning. The strength of each relationship was rated on a scale from 1 (weak) to 5 (strong), guiding the emphasis placed on each pedagogical element throughout the didactic sequence.

Autonomous Learning is strongly supported by self-regulation and self-control and monitoring through scaffolding and the Zone of Proximal Development, with evaluation and feedback playing a crucial role in consolidating geometric reasoning.

Collaborative Learning relies heavily on positive interdependence, tutoring, and collaboration through technological support, which are mediated by adjusted help and shared meaning construction to foster teamwork at Van Hiele Level III.

Problem-Based Learning integrates all socio-constructivist dimensions, particularly in the problematic, investigative/informative, solution-oriented, and productive phases, with strong connections to both Van Hiele levels and the co-construction of knowledge.

The Zone of Proximal Development will be the interactive structure for the development of the self-regulation phase of autonomous learning, given that control of learning is important to recognize the skills of the actual development level and how these are increased in the potential development level with the application of activities guided by the teacher.

The planning phase will have a focus on mediation since it contemplates the creation and planning of digital didactic resources from the didactic sequences for learning the geometric application of notable products.
Learning strategies will be contemplated from scaffolding to contribute significantly to the pedagogical actions planned for knowledge acquisition and decide when implementation is pertinent to achieve educational goals.

Scaffolding will allow the teacher's role to be present, proactively monitoring the educational process, as it leads to recognizing intervention spaces, the design and application of educational actions to achieve learning objectives.

Evaluation of autonomous learning and group evaluation of collaborative learning will be decisive for endorsing the knowledge acquired in the geometric reasoning levels with the traceability of variational thinking to move from one level to another; however, the assessment instruments are designed to encourage discussion spaces between the student and the teacher through the adjusted help phase; the application will be flexible to maintain motivation and fulfillment of the learning objective.

Positive interdependence in collaborative learning will be immersed in the construction of shared meanings, at the third level of geometric reasoning and in the geometric application of notable products with clear academic group objectives for the development of teamwork in carrying out learning activities.

The individual responsibility phase will be developed with mediation; to achieve learning objectives, assuming how the actions of the assigned work affect individual performance and how the fulfillment of collective goals is reflected; since collaborative work promotes the strengthening of cognitive and emotional skills.

Regarding the stimulating interaction phase, it is closely related to the construction of shared meanings since it allows recognizing mathematical methods and reasoning for validation and feedback spaces; giving recognition to maintain motivation in the learning and teaching process.

For strengthening teamwork skills, it is considered pertinent to develop them from scaffolding and at the third level of geometric reasoning so that they are present in the geometric application of notable products based on collective learning and problem-based learning.

Mediation will be present in the skill development phase because the pedagogical design contemplates the creation of learning activities that promote the communication of mathematical ideas and skills to solve mathematical problems.

The problem selection category is determined by scaffolding since the teacher considers their participation important in the exchanges of mathematical resolution ideas to guide and contribute from the mathematical knowledge they possess. Thus, it will be a collective construction in the learning path of the geometric application of notable products, as learning activities will be implemented with digital resources to develop skills in interpreting geometric representations and algebraic procedures.

V. Design and Implementation of the DHA

This research work for teaching the geometric application of notable products to eighth-grade students of the Institución Educativa Augusto Zuluaga in the municipality of Pereira implemented the pedagogical design with a dynamic hypermedial aid through a didactic sequence with the following characteristics:

The DHA is presented in the Cmap Tools program and was constructed for three levels of learning geometric-algebraic concepts (digital resources in PowerPoint and refinements in Thatquiz) and for the geometric application of notable products (digital resources in GeoGebra) with their respective learning guides.

It is available at: https://cmapspublic2.ihmc.us/rid=1YJ1WF21K-1QKL43P-4L77/AHD%20PARA%20LA%20ENSE%C3%91ANZA%20DE%20LOS%20PRODUCTOS%20NOTABLES.cmap

A. Presentation of the Methodological Design

The DHA contemplates the following pedagogical strategies for teaching the geometric application of notable products:

- Diagnosis: Excel resource to assess students' prior knowledge and classify them according to the geometric reasoning levels proposed by Van Hiele.
- Database: Inform the learning level chosen by the student considering the diagnostic report and analysis of results in conjunction with the teacher.
- Learning Level Information: Review the mathematical objectives and competencies related to the learning levels.
- Learning Guide: Didactic resource for the development of learning activities with mathematical concepts and instructions for using the digital resource.
- Teacher Explanation (meet): Interaction for the explanation of mathematical pedagogical content and technology based on the didactic sequence designed for the geometric reasoning levels and the application of notable products.
- Learning Activity: Designed for the acquisition of mathematical knowledge and skills according to Van Hiele's geometric reasoning levels (PowerPoint) and the application of notable products (GeoGebra).
- Feedback: Space for reflection on mathematical ideas, mathematical procedures, and use of the digital resource.
- Refinement and Adjusted Help (only the three learning levels): Activity designed in Thatquiz to assess acquired learning and continue to the next level, with the option of resorting to adjusted help if the refinement results are not correct.
- Padlet: A web space was designed for students to submit all activity results.

Table I
Level 1: recognition or visualization

Level 1: recognition or visualization Autonomous learning	
Competence	Communication and representation
Learning	• Geometric figures • Algebraic language
General objective	Recognizes geometric figures in the contexts they frequent and also makes comparisons expressing them with algebraic language
Specific objectives	• Recognizes geometric figures through visualization • - Understands that two figures of different sizes can be expressed through algebraic language with the use of the didactic resource

Table II
Level 2: analysis

Level 2: analysis Autonomous learning	
Competence	Communication and representation
Learning	• Properties of quadrilaterals • Like terms • Autonomous learning
General objective	Recognize and classify quadrilaterals exclusively by their properties
Specific objectives	• Discover the name of the quadrilateral and group them based on their properties • Establish similarity and difference relationships between two figures with the use of the didactic resource

Table III
Level 3: informal deduction or order

Level 3: informal deduction or order Collaborative learning	
Competence	Communication and representation
Learning	• Area • Multiplication of algebraic expressions • Collaborative Learning
General objective	Perform logical classifications of objects and discover new properties based on properties, and thus understand how to find the area according to the type of given quadrilateral
Specific objectives	• Calculate the area of quadrilaterals from a mathematical problem and use the educational resource • Understand the mathematical process of multiplying algebraic expressions and geometric construction in the socialization of mathematical ideas

Table IV
Geometric application of notable products

Notable products	
Specific objectives	Communication and representation
Learning	• Square of a binomial (sum and difference) • Product of binomials with a common term • Product of sum by difference • - Problem-Based Learning • Collaborative Learning
General objective	Interpret the geometric representations of notable products and solve mathematical problems based on geometric constructions
Specific objectives	• Find the notable product • Construct the graphic representation from the problem and use the didactic resource for resolution

The DHA contemplates learning activities designed from learning theories (autonomous learning, collaborative learning, and problem-based learning) for the geometric application of notable products with reasoning levels (Van Hiele) and algebraic contents.

B. Design of the Didactic Sequence

The didactic sequence was created from the pedagogical design for learning the geometric application of notable products; where eight sequences are defined to be executed in eight hours each, corresponding to diagnosis, levels, and application of notable products.

C. Design of the Learning Guide

For the development of learning activities, the pedagogical design contemplated the implementation of eight guides to provide pedagogical support in the execution of the diagnosis, geometric-variational reasoning levels, and the geometric application of notable products.

VI. Results

The results demonstrate both pedagogical and learning-related outcomes derived from the implementation of DHA.

A. Student Learning Outcomes

Students showed measurable improvements in:

a. Understanding of binomial square through geometric representation
b. Ability to connect algebraic expressions with visual models
c. Problem-solving performance in contextualized tasks
d. For example, 75% of students correctly constructed geometric representations of $(a+b)^2$ after the intervention, compared to 28% in the diagnostic phase.

Qualitative Findings

Observational data revealed increased student engagement, particularly during GeoGebra-based activities. Students demonstrated collaborative problem-solving and verbalization of mathematical reasoning.

Teacher Development

The teacher developed competencies in technological pedagogical content knowledge (TPACK), particularly in designing digital resources aligned with learning objectives.

VII. Discussion

The findings suggest that DHA provides a meaningful advancement over traditional ICT-based approaches by integrating structured pedagogical design with interactive visualization tools. Unlike conventional methods, DHA emphasizes the co-construction of knowledge and adaptive learning pathways aligned with Van Hiele levels.

VIII. Conclusions

This study contributes to mathematics education by proposing and validating a DHA-based pedagogical model that integrates ICT, learning theories, and geometric reasoning. The main contribution lies in demonstrating how structured digital learning environments can enhance both teaching practices and student learning outcomes. Future research should focus on quantitative validation with larger samples, as well as comparative studies incorporating control groups.

The pedagogical design implemented for teaching notable products to eighth-grade students in a virtual learning context—particularly during the pandemic period—required the teacher to adapt her instructional approach by integrating socio-constructivist principles with autonomous, collaborative, and problem-based learning theories. This approach was aligned with the development of geometric reasoning levels (Van Hiele, 2001) and algebraic understanding, enabling students to construct geometric representations of notable products through the use of ICT as a tool for solving contextualized mathematical problems.

This pedagogical process fostered greater dynamism in the interaction between the teaching content, the teacher's subject matter knowledge, and students' learning processes [20]. Furthermore, it supported the effective integration of technological pedagogical content knowledge (TPACK) [21] within a learning environment structured by the proposed design.

The implementation of Dynamic Hypermedial Aids (DHA) and the didactic sequence enabled the teacher to effectively plan the teaching of notable products and select appropriate instructional strategies in alignment with the theoretical framework of the methodological design (diagnosis, learning guides, technological resources, refinement processes, and adjusted help). These elements were embedded within learning pathways structured according to Van Hiele's levels of geometric reasoning and supported by the integration of variational thinking within the DHA environment to facilitate the understanding of geometric applications in problem-solving contexts.

Furthermore, the teacher developed competencies in the use of ICT to innovate and guide the educational process, supporting the effective delivery of mathematical content and clarifying students' understanding of both the purpose and application of technological tools within learning activities. This process also promoted collaborative knowledge construction and encouraged the recognition and integration of students' prior knowledge in the design of didactic sequences.

In addition, the teacher strengthened her ability to manage unforeseen situations during the implementation of the didactic sequence, such as limitations in digital resources, connectivity issues, and external conditions affecting the learning environment. As a result, the teacher enhanced her competencies in Technological Pedagogical Content Knowledge (TPACK) within the context of mathematics teaching.

The application of Dynamic Hypermedial Aids (DHA) in a virtual learning environment for teaching notable products enabled the teacher to create effective feedback spaces for discussing mathematical arguments. It also demonstrated her ability to guide and intervene in students' errors by integrating content knowledge with technological pedagogical knowledge. In addition, the teacher strengthened her capacity to recognize students' actions in the internalization of prior knowledge and to support the subsequent development of mathematical concepts through the didactic strategies outlined in the methodological design. This process contributed to clarifying students' understanding of both the use and purpose of digital resources in learning activities.

Through the implementation of the pedagogical design, grounded in learning theories and a socio-constructivist approach, the teacher successfully adapted the teaching of notable products to a fully virtual environment. In doing so, she developed competencies aligned with the Knowledge Quartet model, particularly within the transformation dimension—by selecting appropriate geometric representations and examples—and the contingency dimension—by effectively responding to unforeseen situations during the implementation of the didactic sequence.

Furthermore, the creation of didactic strategies and multimedia resources within the DHA environment contributed to the development of the teacher's technological knowledge, pedagogical knowledge [22], and specialized content knowledge [20]. This was reflected in the design of structured learning pathways aligned with Van Hiele's three levels of geometric reasoning, as well as with algebraic content and the application of notable products. These elements were integrated into a pedagogical matrix aimed at fostering mathematical understanding and promoting the development of autonomous, collaborative, and problem-based learning skills.

Additionally, the teacher's didactic contribution is evident in the development of a structured approach for understanding geometric representations of notable products as a means of solving mathematical problems. This approach incorporated knowledge of content and students [20], particularly through the diagnostic phase used to classify students' reasoning levels and initiate a more targeted teaching and learning process based on prior knowledge. The diagnostic results were essential for informing the pedagogical design, allowing the teacher to recognize students' mathematical and technological competencies, as well as their individual and collaborative skills, and to align them effectively with instructional objectives.

Declarations

Author Contributions

Conceptualization, L.M.D.M. and J.R.G.G.; Methodology, L.M.D.M.; Validation, L.M.D.M. and J.R.G.G.; Formal Analysis, L.M.D.M.; Investigation, L.M.D.M.; Resources, J.R.G.G. and J.F.A.M.; Data Curation, L.M.D.M.; Writing – Original Draft Preparation, L.M.D.M.; Writing – Review & Editing, J.R.G.G. and J.F.A.M.; Supervision, J.R.G.G. and J.F.A.M.; Project Administration, L.M.D.M. All authors have read and agreed to the published version of the manuscript.

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Ethical review and approval were waived for this study, as it involved standard educational practices and anonymized data analysis in accordance with institutional policies.

Informed Consent Statement

Informed consent was obtained from all subjects involved in the study.

Data Availability Statement

The data presented in this study are available on request from the corresponding author.

Conflicts of Interest

The authors declare no conflict of interest.

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