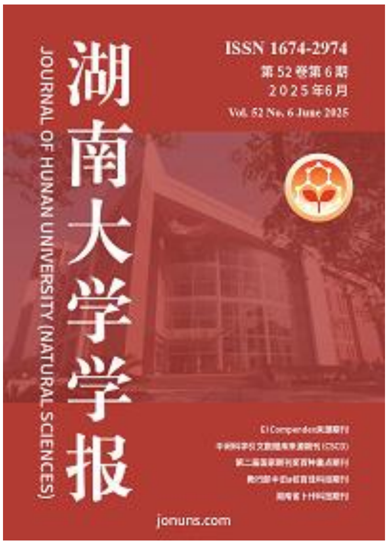

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Development of a Generalized One-Dimensional Hydrodynamic Equation via the Dual Approach

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Abstract: This study aims to develop a more general form of the one-dimensional hydrodynamic equation by employing the dual approach through direct integration of the three-dimensional hydrodynamic equations. Unlike previous formulations that combine two-horizontal and two-vertical components, this method allows for the derivation of a one-dimensional equation that inherently incorporates the effects of geometric deformation of the river cross-section. The proposed equation enhances the classical Saint–Venant model by introducing additional terms whose orders of magnitude are rigorously evaluated relative to the conventional momentum terms. This methodological innovation offers a more comprehensive representation of real-world flow dynamics in open channels. The novelty of this research lies in the direct derivation of a one-dimensional hydrodynamic equation from the full three-dimensional system via the dual approach, bypassing the need for intermediate two-dimensional simplifications and capturing a broader range of hydrodynamic behavior.

Keywords: Classical average; Dual approach; One-dimensional hydrodynamic equation; Three-dimensional hydrodynamic equation; River cross-section deformation, Saint–Venant model.

基于双重方法构建一维水动力学方程

摘要：

本研究旨在通过对三维水动力学方程的直接积分，采用双重方法（Dual Approach）构建一种更为广义的一维水动力学方程。与先前结合水平和垂直二维分量的构建方式不同，本文方法能够在一维建模中本质性地引入河道横断面几何变形的影响。所提出的方程通过引入相对于经典动量项量纲上经过严格评估的新项，拓展并完善了经典的Saint–Venant模型。该方法学创新为开放通道中水流的真实动态行为提供了更全面的数学刻画。本研究的创新之处在于：通过双重方法，直接从完整的三维系统中推导出一维水动力学方程，避免了中间二维简化过程，因而能够捕捉更广泛的水动力特征。

关键词： 经典平均；双重方法；一维水动力学方程；三维水动力学方程；河道断面变形；Saint–Venant模型

1. Introduction

The classical averaged equations governing flow and tracer transport have been extensively presented in the literature, providing foundational tools for modeling hydraulic and environmental processes [1–8]. However, these formulations are typically based on conventional averaging techniques that may not fully capture complex flow behaviors in natural and engineered systems. The dual approach introduces a conceptual framework that simultaneously considers two complementary aspects of a given problem, allowing for a more comprehensive analysis. Notably, dual averaging has been demonstrated to generalize and extend classical averaging methods, offering improved theoretical consistency and broader applicability [9,10]. We applied dual averaging to develop equations for flow and tracer transport in one dimension (1D), two- dimensional horizontal (2DH), two-dimensional vertical (2DV), three dimensions (3D). This approach resulted in new equations and systems of equations that are more general than the classical ones [11-18]. Previously, to obtain the one-dimensional hydrodynamic equations in [17], we combined the two-dimensional horizontal equations [12] and two-dimensional vertical equations from hydrodynamic equations [17], both of which were built using the dual approach GLAV (global-local average value).

In this paper, we develop a one-dimensional hydrodynamic equation directly using the dual approach GLAV [9, 10, 16, 17] from the classical three-dimensional hydrodynamic equations [1-5, 7, 8].

2. Building a one-dimensional hydrodynamic equation directly using the dual approach from the classical three-dimensional hydrodynamic equations

We have the classical three-dimensional hydrodynamic equations or Navier-Stokes equations as follows [1- 4, 7, 8]:

$$\left.\begin{array}{c}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \\
\frac{\partial u}{\partial t}+\frac{\partial (u.u)}{\partial x}+\frac{\partial (u.v)}{\partial y}+\frac{\partial (u.w)}{\partial z}=\frac{1}{\rho}.F_x \\
-\frac{1}{\rho}.\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \\
\frac{\partial v}{\partial t}+\frac{\partial (u.v)}{\partial x}+\frac{\partial (v.v)}{\partial y}+\frac{\partial (v.w)}{\partial z}=\frac{1}{\rho}.F_y - \\
\frac{1}{\rho}.\frac{\partial p}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z} \\
\frac{\partial w}{\partial t}+\frac{\partial (u.w)}{\partial x}+\frac{\partial (v.w)}{\partial y}+\frac{\partial (w.w)}{\partial z}=\frac{1}{\rho}.F_z - \\
\frac{1}{\rho}.\frac{\partial p}{\partial z}+\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}
\end{array}\right\} \qquad (1)$$

Where: *u, v, w* are the components of velocity in the *x, y* and *z* directions; F_x, F_y, F_z are the components of the resultant external force in the *x, y* and *z* directions respectively; *p* is the pressure component; τ_{xx}, τ_{yy}, τ_{zz}, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy} are the stresses (including both molecular and turbulent effects); ρ is the density of water.

Using the dual approach for Equations (1) along the y-axis (Fig. 1), the cross-sectional direction of the river, we obtained the 2DV hydrodynamic equation [17] as follows:

$$\begin{aligned}
&\underbrace{\gamma_{1c}.\frac{\partial}{\partial t}(b^2)}_{[I]} + \underbrace{\beta_{1c}\frac{\partial}{\partial x}[\overline{u}.b^2]}_{[J]} + \underbrace{\delta_{1c}\frac{\partial}{\partial z}[\overline{w}.b^2]}_{[K]} = 0 \\
&\alpha_{1tx}.\underbrace{\frac{\partial}{\partial t}\{\overline{u}.(b^2)\}}_{[LX]} - \alpha_{1tx}.\underbrace{\overline{u}.\left\{\frac{\partial}{\partial t}(b^2)\right\}}_{[MX]} \\
&+\beta_{1x}\underbrace{\frac{\partial}{\partial x}\{(\overline{uu}).(b^2)\}}_{[NX]} - \beta_{1x}\underbrace{(\overline{uu}).\frac{\partial}{\partial x}(b^2)}_{[OX]} + \\
&\delta_{1x}\underbrace{\frac{\partial}{\partial z}\{(\overline{uw}).(b^2)\}}_{[PX]} - \delta_{1x}\underbrace{(\overline{uw}).\frac{\partial}{\partial z}(b^2)}_{[QX]} \\
&= \frac{1}{\rho}.\underbrace{F_x.(b^2)}_{[RX]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial x}.(b^2)}_{[SX]} + \\
&\frac{1}{\rho}\underbrace{div(\vec{\tau})_x.(b^2)}_{[TX]} - \frac{1}{\rho}\underbrace{(\vec{\tau}.\vec{n})_x.(b)}_{[UX]} \\
&\alpha_{1tz}\underbrace{\frac{\partial}{\partial t}\{\overline{w}.(b^2)\}}_{[LZ]} - \alpha_{1tz}.\underbrace{\overline{w}.\left\{\frac{\partial}{\partial t}(b^2)\right\}}_{[MZ]} \\
&+\beta_{1z}\underbrace{\frac{\partial}{\partial x}\{(\overline{wu}).(b^2)\}}_{[NZ]} - \beta_{1z}\underbrace{(\overline{wu}).\frac{\partial}{\partial x}(b^2)}_{[OZ]} + \\
&\delta_{1z}\underbrace{\frac{\partial}{\partial z}\{(\overline{\mathrm{ww}}).(b^2)\}}_{[PZ]} - \delta_{1z}\underbrace{(\overline{ww}).\frac{\partial}{\partial z}(b^2)}_{[QZ]} \\
&= \frac{1}{\rho}.\underbrace{F_z.(b^2)}_{[RZ]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial z}.(b^2)}_{[SZ]} + \\
&\frac{1}{\rho}\underbrace{div(\vec{\tau})_z.(b^2)}_{[TZ]} - \frac{1}{\rho}\underbrace{(\vec{\tau}.\vec{n})_z.(b)}_{[UZ]}
\end{aligned} \tag{2}$$

where:

b is the river width $b = Y_2 - Y_1$; $\overline{u}$ and $\overline{w}$ denote the velocity along the Ox and Oz axis which are averaged over the river width b; F_x, F_z are the components of the resultant external force in the X, and Z directions respectively; p is the pressure; ρ is density of water; $\vec{\tau}$ are the shear stress; $\vec{n}$ is normal vector; $\alpha_{ijk}, \beta_{ij}, \delta_{ij}, \gamma_{ij}$, δ_{ij} are correction coefficients; Their values are close to 1 when the values u, w, uu, uw, ww are replaced with the average values $\overline{u}$, $\overline{w}$, $\overline{uu}$, $\overline{uw}, \overline{\mathrm{ww}}$ along the OY coordinate axis.

For simplicity, we set these coefficients to 1 and ignore the overbars above the quantities u, w, uu, uw, ww to obtain the following 2DV system of equations as follows:

$$\begin{aligned}
&\underbrace{\frac{\partial}{\partial t}(b^2)}_{[I]} + \underbrace{\frac{\partial}{\partial x}[u.b^2]}_{[J]} + \underbrace{\frac{\partial}{\partial z}[w.b^2]}_{[K]} = 0 \\
&\underbrace{\frac{\partial}{\partial t}\{u.(b^2)\}}_{[LX]} - \underbrace{u.\left\{\frac{\partial}{\partial t}(b^2)\right\}}_{[MX]} + \underbrace{\frac{\partial}{\partial x}\{uu.(b^2)\}}_{[NX]} - \underbrace{(uu).\frac{\partial}{\partial x}(b^2)}_{[OX]} \\
&+ \underbrace{\frac{\partial}{\partial z}\{(uw).(b^2)\}}_{[PX]} - \underbrace{(uw).\frac{\partial}{\partial z}(b^2)}_{[QX]} \\
&= \frac{1}{\rho}.\underbrace{F_x.(b^2)}_{[RX]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial x}.(b^2)}_{[SX]} + \frac{1}{\rho}\underbrace{div(\vec{\tau})_x.(b^2)}_{[TX]} - \frac{1}{\rho}\underbrace{(\vec{\tau}.\vec{n})_x.(b)}_{[UX]} \\
&\underbrace{\frac{\partial}{\partial t}\{w.(b^2)\}}_{[LZ]} - \underbrace{w.\left\{\frac{\partial}{\partial t}(b^2)\right\}}_{[MZ]} + \underbrace{\frac{\partial}{\partial x}\{(wu).(b^2)\}}_{[NZ]} - \underbrace{(wu).\frac{\partial}{\partial x}(b^2)}_{[OZ]} \\
&+ \underbrace{\frac{\partial}{\partial z}\{(\mathrm{ww}).(b^2)\}}_{[PZ]} - \underbrace{(\mathrm{ww}).\frac{\partial}{\partial z}(b^2)}_{[QZ]} \\
&= \frac{1}{\rho}.\underbrace{F_z.(b^2)}_{[RZ]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial z}.(b^2)}_{[SZ]} + \frac{1}{\rho}\underbrace{div(\vec{\tau})_z.(b^2)}_{[TZ]} - \frac{1}{\rho}\underbrace{(\vec{\tau}.\vec{n})_z.(b)}_{[UZ]}
\end{aligned} \tag{3}$$

We continue to average Equation (3) in the Z direction using the dual approach GLAV [9, 10, 16, 17] which means that: Integrating all terms of Equation (3) in the Z direction twice (Fig.2): first, through local integration from Z_b to Z_m, and second, through the global integration from Z_b to Z_s.

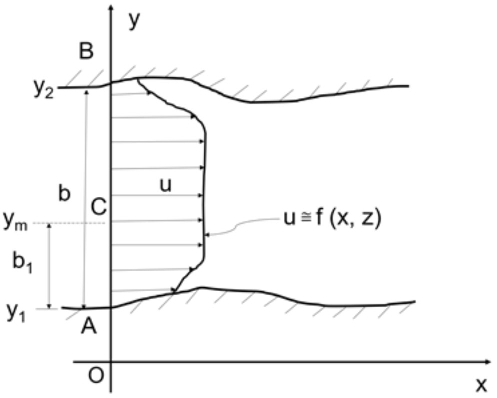

Figure 1. Sketch of the construction of 2DV using the dual approach

The symbols in Figure 1 are as follows:
Z_b is the elevation of the riverbed; Z_m is the elevation corresponding to any depth h_1; Z_s is the water surface elevation; h is the water depth.

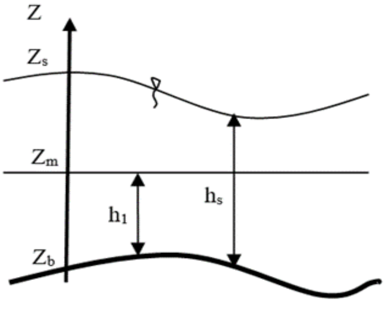

Figure 2. Sketch of the construction of 1D using the dual approach

Integrate all the terms of the continuity equation of the system of equations (3) from Z_b to Z_m for the first time:

$$T[I] = \int_{Z_b}^{Z_m} \frac{\partial}{\partial t}(b^2)dz = \frac{\partial}{\partial t}\int_{Z_b}^{Z_m} b^2 dz - \int_{Z_b}^{Z_m} b^2.\frac{\partial(dz)}{\partial t} = \frac{\partial}{\partial t}(b_1^2)h_1 - b_1^2.\frac{\partial h_1}{\partial t} \tag{4a}$$

Where:

$$b_1 \cong \sqrt{\frac{1}{h_1}\int_{Z_b}^{Z_m} b^2 dz} \cong \sqrt{\frac{1}{\frac{\partial h_1}{\partial t}}\int_{Z_b}^{Z_m} b^2.\frac{\partial(dz)}{\partial t}} \tag{4b}$$

$$T[J] = \int_{Z_b}^{Z_m} \frac{\partial}{\partial x}(ub^2)dz = \frac{\partial}{\partial x}\int_{Z_b}^{Z_m} ub^2 dz - \int_{Z_b}^{Z_m} ub^2.\frac{\partial(dz)}{\partial x} = \frac{\partial}{\partial x}\{(u_1 b_1^2)h_1\} - u_1 b_1^2.\frac{\partial h_1}{\partial x} \tag{4c}$$

Where:

$$u_1 = \frac{1}{h_1 b_1^2}\int_{Z_b}^{Z_m} ub^2 dz \cong \frac{1}{b_1^2.\frac{\partial h_1}{\partial x}}\int_{Z_b}^{Z_m} ub^2.\frac{\partial(dz)}{\partial x}$$

$$T[K] = \int_{Z_b}^{Z_m} \frac{\partial}{\partial z}(wb^2)dz = (wb^2)_m - (wb^2)_b = w_1 b_1^2 - w_b b_b^2 \tag{4d}$$

We obtain the continuity equation [*TCQ*] at any depth *h₁* as follows:

$$[TCQ]: T[I] + T[J] + T[K] = \frac{\partial}{\partial t}(b_1^2)h_1 - b_1^2.\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}\{(u_1 b_1^2)h_1\} - u_1 b_1^2.\frac{\partial h_1}{\partial x} + w_1 b_1^2 - w_b b_b^2 = 0 \tag{5a}$$

$$[TCQ]: T[I] + T[J] + T[K] = \frac{\partial}{\partial t}(b_1^2)h_1 + \frac{\partial}{\partial x}\{(u_1 b_1^2)h_1\} - b_1^2.\underbrace{(\frac{\partial h_1}{\partial t} + u_1.\frac{\partial h_1}{\partial x} - w_1)}_{[TCQ1]} - w_b b_b^2 = 0 \tag{5b}$$

At any point in the vertical two-dimensional flow domain (X, Z), we have the kinematic boundary condition [*TCQ1*] =0; at the bottom of the channel, the vertical velocity $w_b = 0$, so equation (5b) becomes:

$$[TCQ]: T[I] + T[J] + T[K] = \underbrace{\frac{\partial}{\partial t}(b_1^2)h_1}_{[TI]} + \underbrace{\frac{\partial}{\partial x}\{(u_1 b_1^2)h_1\}}_{[TJ]} = 0 \tag{5c}$$

Equation (5c) is the classical continuity equation of the one-dimensional (1D) open flow problem.

Continue integrating, from *Z_b* to *Z_s* , all terms of the continuity equation (5c) a second time.

Calculate the integral of the term [*TI*] of Equation (5c) from Z_b to Z_s a second time:

$$T^2[I] \equiv T[TI] = \int_{Z_b}^{Z_s} \frac{\partial}{\partial t}(b_1^2)h_1 dz = \frac{\partial}{\partial t}\int_{Z_b}^{Z_s} b_1^2 h_1 dz - \int_{Z_b}^{Z_s} b_1^2 h_1 \frac{\partial}{\partial t}(dz)$$

$$T^2[I] = \frac{1}{2}\alpha_1.\frac{\partial}{\partial t}\{b_1^2.(Z_s^2 - Z_b^2\} - \frac{1}{2}\alpha_2.b_1^2.\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) \tag{6a}$$

Where

α_1, α_2 are the correction coefficients, defined similarly to those in equation (2), and are calculated as follows:

$$\alpha_1 = \frac{2}{\frac{\partial}{\partial t}\{b_1^2.(Z_s^2 - Z_b^2\}}\frac{\partial}{\partial t}\int_{Z_b}^{Z_s} b_1^2 h_1 dz;$$

$$\alpha_2 = \frac{2}{b_1^2.\frac{\partial}{\partial t}(Z_s^2 - Z_b^2)}\int_{Z_b}^{Z_s} b_1^2 h_1 \frac{\partial}{\partial t}(dz)$$

Calculate the integral of the term [*TJ*] from Equation (5c) a second time, from Z_b to Z_s :

$$T^2[J] \equiv T[TJ] = \int_{Z_b}^{Z_s} \frac{\partial}{\partial x}\{(u_1 b_1^2)h_1\}dh_1 = \frac{\partial}{\partial x}\int_{Z_b}^{Z_s}(u_1 b_1^2)h_1 dz - \int_{Z_b}^{Z_s} u_1 b_1^2 h_1 \frac{\partial}{\partial x}(dz)$$

$$T^2[J] = \frac{1}{2}\alpha_3.\frac{\partial}{\partial x}\{ub^2.(Z_s^2 - Z_b^2\} - \frac{1}{2}\alpha_4.ub^2.\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) \tag{6b}$$

Where:

α_3, α_4 are the correction coefficients, defined similarly to those in equation (2), and are calculated as follows:

$$\alpha_3 = \frac{2}{\frac{\partial}{\partial x}\{ub^2.(Z_s^2 - Z_b^2\}}\frac{\partial}{\partial x}\int_{Z_b}^{Z_s} u_1 b_1^2 h_1 dz;$$

$$\alpha_4 = \frac{2}{ub^2.\frac{\partial}{\partial x}(Z_s^2 - Z_b^2)}\int_{Z_b}^{Z_s} u_1 b_1^2 h_1 \frac{\partial}{\partial t}(dz).$$

Therefore, we have:

$$[T^2CQ]: T^2[I] + T^2[J] = \frac{1}{2}\alpha_1.\frac{\partial}{\partial t}\{b_1^2.(Z_s^2 - Z_b^2)\} - \frac{1}{2}\alpha_2.b_1^2.\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) + \frac{1}{2}\alpha_3.\frac{\partial}{\partial x}\{ub^2.(Z_s^2 - Z_b^2\} - \frac{1}{2}\alpha_4.ub^2.\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) = 0 \tag{7a}$$

Simplifying and adding the term bw_s to both sides of equation (7a), we obtain the continuity equation established by the dual approach:

$$[T^2CQ]: \frac{\partial}{\partial t}(Ab) + \frac{\partial}{\partial x}(Qb) - b^2(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - w_s) = b^2 w_s \tag{7b}$$

Or:

$$[T^2CQ]: \frac{\partial}{\partial t}(Ab) + \frac{\partial}{\partial x}(Qb) = b^2 w_s \tag{7c}$$

Integrating all terms of the equation of motion in the OX direction in the system of equations (3), for the first time, we have:

$$T[LX] = \int_{Z_b}^{Z_m} \frac{\partial}{\partial t}(ub^2)dz = \frac{\partial}{\partial t}\int_{Z_b}^{Z_m} ub^2 dz - \int_{Z_b}^{Z_m} ub^2.\frac{\partial(dz)}{\partial t} = \frac{\partial}{\partial t}(u_1 b_1^2)h_1 - u_1 b_1^2.\frac{\partial h_1}{\partial t} \tag{8a}$$

$$T[MX] = \int_{Z_b}^{Z_m} u\frac{\partial}{\partial t}(b^2)dz = u_1\frac{\partial}{\partial t}(b_1^2)(Z_m - Z_b) = u_1\frac{\partial}{\partial t}(b_1^2)h_1 \tag{8b}$$

$$T[NX] = \int_{Z_b}^{Z_m} \frac{\partial}{\partial x}\{(uu).(b^2)\}dz = \frac{\partial}{\partial x}\int_{Z_b}^{Z_m}\{(uu).(b^2)\}dz - \int_{Z_b}^{Z_m} uub^2.\frac{\partial(dz)}{\partial x}$$

$$T[NX] = \frac{\partial}{\partial x}\{(u_1 u_1)b_1^2 h_1\} - (u_1 u_1)b_1^2.\frac{\partial h_1}{\partial x} \tag{8c}$$

$$T[OX] = \int_{Z_b}^{Z_m} uu\frac{\partial}{\partial x}(b^2)dz = (u_1 u_1).\frac{\partial b_1^2}{\partial x}(Z_m - Z_b) = (u_1 u_1).\frac{\partial b_1^2}{\partial x}h_1 \tag{8d}$$

$$T[PX] = \int_{Z_b}^{Z_m} F_x(b^2)dz = F_x.b_1^2.(Z_m - Z_b) = F_x.b_1^2.h_1 \tag{8e}$$

$$T[QX] = \int_{Z_b}^{Z_m} \frac{\partial p}{\partial x}.b^2 dz = b_1^2.\left\{\frac{\partial}{\partial x}\int_{Z_b}^{Z_m} pdz - \left(p.\frac{\partial z}{\partial x}\right)_{zm} + \left(p.\frac{\partial z}{\partial x}\right)_{zb}\right\} \tag{8f}$$

To be able to calculate the pressure p in equation (8f), we use the equation of motion in the OZ direction from the system of equations (3):

$$\underbrace{\frac{\partial}{\partial t}\{\overline{w}.(b^2)\}}_{[LZ]} - \underbrace{\overline{w}.\left\{\frac{\partial}{\partial t}(b^2)\right\}}_{[MZ]} + \underbrace{\frac{\partial}{\partial x}\{(\overline{wu}).(b^2)\}}_{[NZ]} - \underbrace{(\overline{wu}).\frac{\partial}{\partial x}(b^2)}_{[OZ]} + \underbrace{\frac{\partial}{\partial z}\{(\overline{\mathrm{ww}}).(b^2)\}}_{[PZ]} - \underbrace{(\overline{ww}).\frac{\partial}{\partial z}(b^2)}_{[QZ]} = \frac{1}{\rho}.\underbrace{F_z.(b^2)}_{[RZ]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial z}.(b^2)}_{[SZ]} + \frac{1}{\rho}\underbrace{div(\vec{\tau})_z.(b^2)}_{[TZ]} - \frac{1}{\rho}\underbrace{(\vec{\tau}.\vec{n})_z.(b)}_{[UZ]}$$

In the case of 1D open flow, the velocity in the OZ direction is negligible ($w_b \approx 0$), so we ignore the terms *[LZ], [MZ], [NZ], [OZ], [PZ], [QZ], [TZ], [UZ]*; therefore, this equation is rewritten as follows:

$$\frac{1}{\rho}.\underbrace{F_z.(b^2)}_{[RZ]} - \frac{1}{\rho}.\underbrace{\frac{\partial p}{\partial z}.(b^2)}_{[SZ]} = 0 \tag{8g}$$

Consider the fluid flowing in a gravity field, where the external force $F_z = -\rho g$; we see that equation (8g) is the basic differential equation of hydrostatics [19-21] and we know that the pressure p at any point in the flow domain is given by following:

$$p = \rho g(Z_s - Z_m) \tag{8h}$$

Substituting the pressure p from equation (8h) into equation (8f), we get:

$$T[QX] = \rho g.b_1^2.h_{1.}\frac{\partial Z_s}{\partial x} \tag{8i}$$

We continue calculating the terms in the equation of motion along the OX axis of the system of equations (3):

$$T[RX] = \int_{Z_b}^{Z_m} div(\vec{\tau})_x.(b^2)dz = div(\vec{\tau})_x.b_1^2.(Z_m - Z_b) = div(\vec{\tau})_x.b_1^2.h_1 \tag{8j}$$

$$T[SX] = \int_{Z_b}^{Z_m}(\vec{\tau}.\vec{n})_x.(b)\,dz = (\vec{\tau}.\vec{n})_x.(b_1).(Z_m - Z_b) = (\vec{\tau}.\vec{n})_x.(b_1).h_1 \tag{8k}$$

$$T[T] = \int_{Z_b}^{Z_m}\frac{\partial}{\partial z}\{(uw).(b^2)\}\,dz = \{(uw).(b^2)\}_m - \{(uw).(b^2)\}_b \tag{8l}$$

$$T[U] = \int_{Z_b}^{Z_m}(\overline{uw}).\frac{\partial}{\partial z}(b^2)\,dz = (uw)_m(b_m^2 - b_b^2) \tag{8m}$$

Summing the first integrated terms of the equation of motion along the OX axis, we obtain:

$$[\mathrm{TME(ox)}]: \frac{\partial}{\partial t}(u_1 b_1^2)h_1 - u_1\frac{\partial}{\partial t}(b_1^2)h_1 + \frac{\partial}{\partial x}\{(u_1u_1b_1^2)h_1\} + (u_1u_1).\frac{\partial b_1^2}{\partial x}h_1 - (ub^2)_m\left(\frac{\partial h_1}{\partial t} + u_1.\frac{\partial h_1}{\partial x} - w_1\right) = \frac{1}{\rho}F_x b_1^2.h_1 - \mathrm{g}b_1^2.h_{1.}\frac{\partial Z_s}{\partial x} + \frac{1}{\rho}div(\vec{\tau})_x.b_1^2.h_1 - \frac{1}{\rho}(\vec{\tau}.\vec{n})_x.(b_1).h_1 \tag{9a}$$

$$\mathrm{TME[ox]}: \underbrace{\frac{\partial}{\partial t}(u_1 b_1^2)h_1}_{T[LX]} - \underbrace{u_1\frac{\partial}{\partial t}(b_1^2)h_1}_{T[MX]} + \underbrace{\frac{\partial}{\partial x}\{(u_1u_1b_1^2)h_1\}}_{T[NX]} + \underbrace{u_1u_1.\frac{\partial b_1^2}{\partial x}h_1}_{T[OX]} = \underbrace{\frac{1}{\rho}F_x b_1^2.h_1}_{T[PX]} - \underbrace{gb_1^2.h_{1.}\frac{\partial Z_s}{\partial x}}_{T[QX]} + \underbrace{\frac{1}{\rho}div(\vec{\tau})_x.b_1^2.h_1}_{T[RX]} - \underbrace{\frac{1}{\rho}(\vec{\tau}.\vec{n})_x.(b_1).h_1}_{T[SX]} \tag{9b}$$

Integrating the terms of the equation of motion in the system of equations (3) in the direction OX for the second time, that is, integrating the terms of equation (9b), we get:

$$T^2[LX] = \int_{Z_b}^{Z_s}\frac{\partial}{\partial t}(u_1b_1^2)h_1dz = \underbrace{\frac{\partial}{\partial t}\int_{Z_b}^{Z_s}(u_1b_1^2)h_1dz}_{T^2[LX1]} - \underbrace{\int_{Z_b}^{Z_s}(u_1b_1^2)h_1.\frac{\partial(dz)}{\partial t}}_{T^2[LX2]}$$

$$T^2[LX1] = \frac{\partial}{\partial t}\int_{Z_b}^{Z_s}(u_1b_1^2)h_1dz = \frac{1}{2}\beta_1\frac{\partial}{\partial t}\left\{\left(\overline{ub^2}\right)\left(Z_s^2 - Z_b^2\right)\right\}$$

$$T^2[LX2] = \int_{Z_b}^{Z_s}(u_1b_1^2)h_1.\frac{\partial(dz)}{\partial t} = \frac{1}{2}\beta_1\left(\overline{ub^2}\right)\frac{\partial}{\partial t}\left(Z_s^2 - Z_b^2\right)$$

So, we have:

$$T^2[LX] = \frac{1}{2}\beta_1\frac{\partial}{\partial t}\left\{\left(\overline{ub^2}\right)\left(Z_s^2 - Z_b^2\right)\right\} - \frac{1}{2}\beta_1\left(\overline{ub^2}\right)\frac{\partial}{\partial t}\left(Z_s^2 - Z_b^2\right) \tag{10a}$$

where:

$$\beta_1 = \frac{2}{\frac{\partial}{\partial t}\left\{\left(\overline{ub^2}\right)\left(Z_s^2 - Z_b^2\right)\right\}}.$$

$$\frac{\partial}{\partial t}\int_{Z_b}^{Z_s}(u_1b_1^2)h_1dz \cong \frac{2}{\left(\overline{ub^2}\right)\frac{\partial}{\partial t}\left(Z_s^2 - Z_b^2\right)}\int_{Z_b}^{Z_s}(u_1b_1^2)h_1.\frac{\partial(dz)}{\partial t}$$

$$T^2[MX] = \int_{Z_b}^{Z_s}u_1\frac{\partial}{\partial t}(b_1^2)h_1dz = \frac{1}{2}\gamma_1\overline{u}.\frac{\partial\overline{b^2}}{\partial t}.(Z_s^2 - Z_b^2) \tag{10b}$$

$$\gamma_1 = \frac{2}{\overline{u}..\frac{\partial\overline{b^2}}{\partial t}.\left(Z_s^2 - Z_b^2\right)}.\int_{Z_b}^{Z_s}u_1.\frac{\partial b_1^2}{\partial t}h_1dz$$

$$T^2[NX] = \int_{Z_b}^{Z_s}\frac{\partial}{\partial x}\{(u_1u_1)b_1^2h_1\}dz = \underbrace{\frac{\partial}{\partial x}\int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1dz}_{T^2[NX1]} - \underbrace{\int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1.\frac{\partial(dz)}{\partial x}}_{T^2[NX2]}$$

$$T^2[NX1] = \frac{\partial}{\partial x}\int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1dz = \frac{1}{2}\delta_1\frac{\partial}{\partial x}\left\{(\overline{uu}).\overline{b^2}.\left(Z_s^2 - Z_b^2\right)\right\}$$

$$T^2[NX2] = \int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1.\frac{\partial(dz)}{\partial x} = \frac{1}{2}\delta_1(\overline{uu}).\overline{b^2}\frac{\partial}{\partial x}\left(Z_s^2 - Z_b^2\right)$$

$$T^2[NX] = T^2[NX1] - T^2[NX2] = \frac{1}{2}\delta_1\frac{\partial}{\partial x}\left\{(\overline{uu}).\overline{b^2}.\left(Z_s^2 - Z_b^2\right)\right\} - \frac{1}{2}\delta_1(\overline{uu}).\overline{b^2}\frac{\partial}{\partial x}\left(Z_s^2 - Z_b^2\right) \tag{10c}$$

where:

$$\delta_1 = \frac{2}{\frac{\partial}{\partial x}\left\{(\overline{uu}).\overline{b^2}.\left(Z_s^2 - Z_b^2\right)\right\}}.$$

$$\frac{\partial}{\partial x}\int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1dz \cong \frac{2}{(\overline{uu}).\overline{b^2}\frac{\partial}{\partial x}\left(Z_s^2 - Z_b^2\right)}\int_{Z_b}^{Z_s}(u_1u_1b_1^2)h_1.\frac{\partial(dz)}{\partial x}$$

$$T^2[OX] = \int_{Z_b}^{Z_s}u_1u_1.\frac{\partial b_1^2}{\partial x}h_1dz = \frac{1}{2}\delta_2(\overline{uu}).\frac{\partial\overline{b^2}}{\partial x}.\left(Z_s^2 - Z_b^2\right) \tag{10d}$$

where:

$$\delta_2 = \frac{2}{(\overline{uu}).\frac{\partial\overline{b^2}}{\partial x}.\left(Z_s^2 - Z_b^2\right)}.\int_{Z_b}^{Z_s}u_1u_1.\frac{\partial b_1^2}{\partial x}h_1dz$$

$$T^2[PX] = \int_{Z_b}^{Z_s}\frac{1}{\rho}F_xb_1^2.h_1dz = \frac{1}{2\rho}\delta_3.F_x.\overline{b^2}.(Z_s^2 - Z_b^2) \tag{10e}$$

where:

$$\delta_3 = \frac{2}{\overline{b^2}.F_x.(Z_s^2 - Z_b^2)}.\int_{Z_b}^{Z_s} F_x b_1^2.h_1 dz$$

$$T^2[QX] = \int_{Z_b}^{Z_s} gb_1^2.\frac{\partial Z_s}{\partial x}.h_{1.}dz = \delta_4 \frac{g\overline{b^2}}{2}.\frac{\partial Z_s}{\partial x}.(Z_s^2 - Z_b^2) \quad (10f)$$

where:

$$\delta_4 = \frac{2}{g\overline{b^2}.(Z_s^2 - Z_b^2)}.\int_{Z_b}^{Z_s} gb_1^2.h_{1.}dz$$

$$T^2[RX] = \int_{Z_b}^{Z_s} \frac{1}{\rho} div(\vec{\tau})_x.b_1^2.h_1.dz = \frac{1}{2\rho}\delta_5\,\overline{b^2}.div(\vec{\tau})_x.\left(Z_s^2 - Z_b^2\right) \quad (10g)$$

where:

$$\delta_5 = \frac{2}{\overline{b^2}.div(\vec{\tau})_x.\left(Z_s^2 - Z_b^2\right)}.\int_{Z_b}^{Z_s} div(\vec{\tau})_x.b_1^2.h_1.dz$$

$$T^2[SX] = \frac{1}{\rho}(\vec{\tau}.\vec{n})_x.(b_1).h_1 = \int_{Z_b}^{Z_s} \frac{1}{\rho}(\vec{\tau}.\vec{n})_x.(b_1).h_1.dz = \frac{1}{2\rho}\delta_6.\overline{b}.(\vec{\tau}.\vec{n})_x.(Z_s^2 - Z_b^2) \quad (10h)$$

$$\delta_6 = \frac{2}{\overline{b}.(\vec{\tau}.\vec{n})_x.(Z_s^2 - Z_b^2)}.\int_{Z_b}^{Z_s} (\vec{\tau}.\vec{n})_x.(b_1).h_1.dz$$

Summing the terms of the equation of motion in the direction OX in the system of equations (3), after integrating the second time, we have:

$$[T^2ME(ox)]:\ T^2[LX] + T^2[MX] + T^2[NX] + T^2[OX] = T^2[PX] + T^2[QX] + T^2[RX] + T^2[SX]$$

$$\frac{1}{2}\beta_1 \frac{\partial}{\partial t}\left\{\left(\overline{ub^2}\right)\left(Z_s^2 - Z_b^2\right)\right\} - \frac{1}{2}\beta_1\left(\overline{ub^2}\right)\frac{\partial}{\partial t}\left(Z_s^2 - Z_b^2\right) - \frac{1}{2}\gamma_1\bar{u}\frac{\partial\overline{b^2}}{\partial t}.\left(Z_s^2 - Z_b^2\right) + \frac{1}{2}\delta_1\frac{\partial}{\partial x}\left\{(\overline{uu}).\overline{b^2}.\left(Z_s^2 - Z_b^2\right)\right\} - \frac{1}{2}\delta_1(\overline{uu}).\overline{b^2}\frac{\partial}{\partial x}\left(Z_s^2 - Z_b^2\right) + \frac{1}{2}\delta_2(\overline{uu}).\frac{\partial\overline{b^2}}{\partial x}.\left(Z_s^2 - Z_b^2\right) = \frac{1}{2\rho}\delta_3.F_x.\overline{b^2}.\left(Z_s^2 - Z_b^2\right) - \delta_4.\frac{g\overline{b^2}}{2}.\frac{\partial Z_s}{\partial x}.\left(Z_s^2 - Z_b^2\right) + \frac{1}{2\rho}\delta_5.\overline{b^2}.div(\vec{\tau})_x.\left(Z_s^2 - Z_b^2\right) - \frac{1}{2\rho}\delta_6.\overline{b}.(\vec{\tau}.\vec{n})_x.\left(Z_s^2 - Z_b^2\right) \quad (11a)$$

Simplifying equation (11a) with attention to the flow depth h=$Z_s - Z_b$, area A = b.h, discharge Q = u.A, we obtain the equation of motion along the OX axis after averaging using the dual approach, as follows:

$$\beta_1.\underbrace{\frac{\partial}{\partial t}(Q)}_{[ME1]} - \beta_2\underbrace{.u\frac{\partial}{\partial t}(A)}_{[ME2]} + \delta_1\underbrace{.\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)}_{[ME3]} - \delta_2\underbrace{.u^2.\frac{\partial}{\partial x}(A)}_{[ME4]} = \underbrace{\frac{\delta_3}{\rho}F_{x.}A}_{[ME5]} - \delta_4\underbrace{.gA\frac{\partial Z_s}{\partial x}}_{[ME6]} \underbrace{+\frac{\delta_5}{\rho}.(A).div(\vec{\tau})_x}_{[ME7]} \quad \underbrace{-\frac{\delta_6}{\rho}.(h).(\vec{\tau}.\vec{n})_x}_{[ME8]} \quad (11b)$$

For simplicity, we set the coefficients $\beta_i, \delta_i \cong 1$ and transform the terms [*ME7]* and [*ME8]* with the following note:

$(\vec{\tau})_x = \mu.\frac{\partial u}{\partial x}$, the term [*ME7]* is the friction term due to the viscous force of the fluid.

On the other hand, we have: τ=ρgRJ ; where R is the hydraulic radius, J is the energy gradient of the flow.

When: b>>h then: R≈h ; therefore, the term [*ME8]* represents the friction term at the bottom of the channel; the friction gradient: $S_f = \frac{Q.|Q|}{K^2}$; ; where: $K = A.\frac{R^{2/3}}{n}$, n is the roughness coefficient of the channel.

We can rewrite equation (11b) as follows:

$$\underbrace{\frac{\partial}{\partial t}(Q)}_{[ME1]} - \underbrace{u\frac{\partial}{\partial t}(A)}_{[ME2]} + \underbrace{\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)}_{[ME3]} - \underbrace{u^2.\frac{\partial}{\partial x}(A)}_{[ME4]} = \underbrace{\frac{1}{\rho}F_{x.}A}_{[ME5]} - \underbrace{gA\frac{\partial Z_s}{\partial x}}_{[ME6]} + \underbrace{\frac{\mu}{\rho}.\frac{\partial^2}{\partial x^2}(Q)}_{[ME7]} - \underbrace{g.A.S_f}_{[ME8]} \quad (11c)$$

Thus, the one-dimensional hydrodynamic equations are obtained by a direct dual approach from the three-dimensional Navier-Stokes equations, meaning that the dual averaging of this system is performed first along the width of the river and then along the vertical direction of the river, as follows:

$$\left.\begin{array}{l} \frac{\partial}{\partial t}(Ab) + \frac{\partial}{\partial x}(Qb) = b^2 w_s \\ \underbrace{\frac{\partial}{\partial t}(Q)}_{[ME1]} - \underbrace{u\frac{\partial}{\partial t}(A)}_{[ME2]} + \underbrace{\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)}_{[ME3]} - \underbrace{u^2.\frac{\partial}{\partial x}(A)}_{[ME4]} = \\ \underbrace{\frac{1}{\rho}F_{x.}A}_{[ME5]} - \underbrace{gA\frac{\partial Z_s}{\partial x}}_{[ME6]} + \underbrace{\frac{\mu}{\rho}.\frac{\partial^2}{\partial x^2}(Q)}_{[ME7]} - \underbrace{g.A.S_f}_{[ME8]} \end{array}\right\} \quad (12)$$

3. Evaluate the order of the terms

The momentum equation of the system of equations (12) has two new terms *[ME2]* and *[ME4]* compared to the classical 1D Saint-Venant equation system; we evaluate the order of these terms in relation to the other terms of this momentum equation.

The data used to evaluate the level here are flood discharge measurements taken on November 29, 30, and 31, year 1983, downstream of the Ve River in Mo Duc district, Quang Ngai province [22].

(i) To compare the order of the terms $[ME1] = \frac{\partial}{\partial t}(Q)$ and $[ME2] = u\frac{\partial}{\partial t}(A)$, we use the rising flood discharge data: ΔQ=860m3/s, corresponding to the time period Δt=86400 sec, average flow velocity u = 3m/s, riverbed cross-section change ΔA=240m2.

We have:

$$[ME1] \cong \frac{\Delta Q}{\Delta t} = \frac{860}{86400} = 0.009$$

$$[ME2] \cong u\frac{\Delta A}{\Delta t} = 3 \times \frac{240}{86400} = 0.003$$

Thus, we see that the terms *[ME1]* and *[ME2]* are of the same order.

(ii) Comparing the order of the terms $[ME3] = \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)$and $[ME4] = u^2.\frac{\partial}{\partial x}(A)$, we use the data of the average flood discharge: Q = 700m3/s, Δx=500m, Δu=0.50 m/s, ΔQ = 50m3/s, ΔA = 72m2.

We have:

$$[\mathrm{ME3}] = \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) \cong Q.\frac{\partial u}{\partial x} + u.\frac{\partial Q}{\partial x} \approx Q.\frac{\Delta u}{\Delta x} + u.\frac{\Delta Q}{\Delta x}$$

$$= 700 \times \frac{0.50}{500} + 3 \times \frac{50}{500} = 1.00$$

$$[ME4] = u^2.\frac{\partial}{\partial x}(A) \cong u^2.\frac{\Delta A}{\Delta x} = 3^2.\frac{72}{500} = 1.30$$

Thus, we see that the terms *[ME3]* and *[ME4]* are of the same order.

By evaluating the order of the new terms *[ME2]* and *[ME4]*, we see that they have the same order as the terms in the classical 1D Saint-Venant equation; therefore, the terms *[ME2]* and *[ME4]* need to be retained in the momentum equation of the system of equations (12).

Comments:

It is evident that the one-dimensional hydrodynamic equations system (12) is more general than equations (28) or (29) in [17], thanks to the presence of two additional terms $[ME2]$ and $[ME4]$.

When cross-section $A \approx Const$, over time and space, then $[ME2] \simeq 0$ and $[ME4] \simeq 0$.

This means that when the cross-sectional geometry of the river changes insignificantly over time and space, we obtain the one-dimensional hydrodynamic equations system as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(Ab) + \frac{\partial}{\partial x}(Qb) = b^2 w_s \\ &\underbrace{\frac{\partial}{\partial t}(Q)}_{[ME1]} + \underbrace{\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)}_{[ME3]} = \underbrace{\frac{1}{\rho}F_{x}.A}_{[ME5]} - \underbrace{gA\frac{\partial Z_s}{\partial x}}_{[ME6]} \\ &\qquad + \underbrace{\frac{\mu}{\rho}.\frac{\partial^2}{\partial x^2}(Q)}_{[ME7]} - \underbrace{g.A.S_f}_{[ME8]} \end{aligned}\right\} \quad (13)$$

Then equation (13) becomes the classical one-dimensional hydrodynamic equation system or Saint-Venant equation [3, 4, 7, 8].

5. Conclusion

In this study, a new one-dimensional hydrodynamic equation for riverine flow was developed using the dual approach by directly averaging the three-dimensional Navier–Stokes equations over the width and depth of the river cross-section. Unlike previous dual-approach formulations, which separately combine two-horizontal and two-vertical flow components, the proposed method allows for a unified derivation. The resulting equation includes two additional terms that explicitly account for temporal and spatial variations in the geometry of the river cross-section, making it more general than traditional Saint–Venant-type models.

This extended formulation represents a significant theoretical contribution, as it bridges the gap between classical one-dimensional models and the more comprehensive three-dimensional hydrodynamic reality. By incorporating geometric deformation effects into a one-dimensional framework, the model retains analytical tractability while improving physical realism. From a practical perspective, the proposed equation enhances the capability to simulate river morphodynamics, including sediment transport, erosion, and deposition processes, which are critical for river engineering, environmental assessment, and infrastructure planning. However, this study is not without limitations. The derivation assumes a sufficiently smooth and continuous variation in channel geometry, and turbulence modeling is limited by the averaging procedure. Moreover, the computational implementation and validation of the proposed equation against field or laboratory data were beyond the scope of this paper.

Future research should focus on:

- Empirical validation of the model through comparison with experimental and observed river flow data;
- Extension of the framework to include non-hydrostatic pressure effects and more detailed sediment transport dynamics;
- Development of efficient numerical schemes for solving the derived equation under real-world conditions;
- Integration with GIS-based river morphology datasets for practical applications in flood forecasting and river training.

By advancing the mathematical foundation of river hydraulics through the dual approach, this work provides a robust basis for future developments in hydrodynamic modeling and water resources engineering.

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