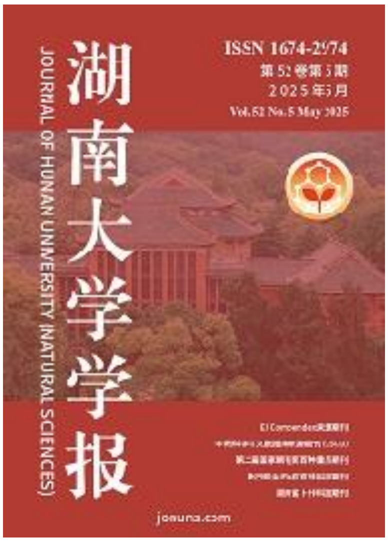

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Exact Solitary-Wave Solutions of the Camassa–Holm Hierarchy via the Sine–Cosine Method

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Abstract: The Sine–Cosine method has emerged as a robust analytical approach to derive solitary wave solutions for nonlinear dispersive partial differential equations. In this work, we systematically employ this technique to the Camassa–Holm hierarchy, encompassing the classical Camassa–Holm, Degasperis–Procesi, Fornberg–Whitham, and Fuchssteiner–Fokas–Camassa–Holm equations. Each member of the hierarchy models shallow-water wave propagation under specific integrability conditions, exhibiting rich dynamical behavior. By applying an appropriate wave transformation, we reduce the governing equations to ordinary differential equations and construct exact travelling-wave solutions in terms of trigonometric and hyperbolic functions. The solutions obtained include compacton like profiles and classical sech^2 and $\cosh^2$ structures, with explicit expressions for wave speed and amplitude as functions of model parameters. Comparative analysis highlights the effectiveness and simplicity of the Sine–Cosine method relative to more elaborate techniques such as the Hirota bilinear formalism and the inverse scattering transform. Our contributions lie in the unified application of this method across the entire hierarchy and the presentation of a comprehensive classification of solitary wave families.

These results extend existing literature and provide valuable benchmarks for numerical simulations and theoretical investigations of nonlinear wave dynamics in fluid mechanics and related fields. Studies illustrate the influence of nonlinearity and dispersion coefficients on wave morphology.

Keywords: Camassa–Holm hierarchy; Sine–Cosine method; solitary wave solutions; nonlinear dispersive equations; integrable systems; exact analytical solutions.

通过正弦–余弦方法求解卡马萨–霍尔姆方程族的精确孤波解

摘要：

正弦–余弦方法已成为一种强有力的解析手段，用于导出非线性色散偏微分方程的孤波解。本文系统地将该方法应用于卡马萨–霍尔姆方程族，包括经典的卡马萨–霍尔姆（CH）方程、Degasperis–Procesi（DP）方程、Fornberg–Whitham（FW）方程及Fuchssteiner–Fokas–Camassa–Holm（FFCH）方程。该族中每个方程均在特定可积条件下描述浅水波的传播，并表现出丰富的动力学特性。通过适当的行波变换，我们将原始偏微分方程化为常微分方程，并构造出以三角函数和双曲函数表示的精确行波解。所得解包括紧支波（compacton）型剖面以及经典的 \sech2\sech^2\sech2 和 cosh2\cosh^2cosh2 结构，并给出了波速与振幅关于模型参数的显式表达式。对比分析表明，相较于 Hirota 双线性方法和反散射变换，正弦–余弦方法具有更高的简便性与效率。本研究的贡献在于该方法在整个方程族中的统一应用与对孤波族的全面分类。

上述结果丰富了现有文献，并为流体力学及相关领域中非线性波动力学的数值模拟与理论研究提供了有价值的参考基准。研究还阐明了非线性系数与色散系数对波形特征的影响。

关键词：

卡马萨–霍尔姆方程族；正弦–余弦方法；孤波解；非线性色散方程；可积系统；精确解析解。

1. Introduction

Solitary waves or solitons are spatially localized, nonlinear disturbances that propagate without changing shape or speed due to an exact balance between nonlinearity and dispersion. Drazin and Johnson [1] provide a comprehensive overview of soliton theory, and Russell's original "wave of translation" observation in 1834 laid the groundwork for mathematical models such as the Korteweg–de Vries (KdV) equation [2].

Ablowitz and Segur [3] demonstrated that many of these models are completely integrable—possessing infinite conservation laws and elastic collision properties—while Remoissenet [4] reviewed their wide-ranging applications from fluid dynamics to optical communications.

Within the Camassa–Holm hierarchy, which includes the Camassa–Holm (CH) equation [17], the Degasperis–Procesi (DP) equation [18], the Fornberg–Whitham (FW) equation, and the Fuchssteiner–Fokas–Camassa–Holm (FFCH) model [20, 21], several gaps persist. First, exact-solution techniques such as the Inverse Scattering Transform [6] and Hirota's bilinear method [5] require intricate spectral analyses that obscure explicit parameter dependencies. Second, although the Sine–Cosine method has been applied successfully to other nonlinear PDEs see Hassan and Mohamad [7] for the classical Boussinesq equation and Yosufoglu and Bekir [8] for coupled systems its systematic use across all CH-hierarchy equations remains unexplored. Third, existing studies treat each equation in isolation, lacking a unified classification of solution families and a cohesive parametric analysis of amplitude and speed.

Recent advances have extended analytical methods to *variable-coefficient* nonlinear equations: Güner [9] demonstrated the Sine–Cosine method for time-dependent coefficient models, and Güner and Bekir [10] obtained exact travelling-wave solutions for similar systems. These developments highlight the method's flexibility but stop short of a comprehensive treatment of the CH hierarchy.

This study employs the Sine–Cosine method a direct symbolic technique that transforms each PDE into an ordinary differential equation via a travelling-wave ansatz and balances the highest-order nonlinear and derivative terms to:

1. Derive exact travelling-wave solutions for the CH, DP, FW, and FFCH equations in a unified framework.
2. Classify solution families into standard functional profiles.
3. Quantify the influence of dispersion and nonlinearity coefficients on wave speed and amplitude.
4. Compare transparency and computational efficiency against the IST and Hirota methods.

By addressing these questions, we establish a coherent analytical framework for solitary-wave solutions in nonlinear dispersive systems, offering benchmarks for theoretical, numerical, and experimental research in fluid mechanics and beyond.

To set the stage for applying the Sine–Cosine method in a unified way, we begin with a general form of a nonlinear dispersive evolution equation:

$$P(u, u_t, u_x, u^n u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (1)$$

where $u(x,t)$ represents the wave profile.

The method begins by applying a traveling wave transformation $\xi = x - ct$, so that $u(x,t) = u(\xi)$. This reduces the PDE (1) to an ordinary differential equation (ODE):

$$P\left(u, u_\xi, u^n u_\xi, u_{\xi\xi}, \dots\right) = 0, \quad (2)$$

with derivatives transforming as follows:

$$\frac{\partial}{\partial t} = -c\frac{d}{d\xi}, \qquad \frac{\partial^2}{\partial t^2} = c^2\frac{d^2}{d\xi^2}, \qquad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \qquad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}. \quad (3)$$

Next, we assume a solution of the form:

$$u(\xi) = \lambda \sin^\beta(\mu\xi), \text{ for } |\xi| \le \frac{\pi}{\mu}, \quad (4)$$

and $u(\xi) = 0$, otherwise.

Or alternatively,

$$u(\xi) = \lambda \cos^\beta(\mu\xi), \text{ for } |\xi| \le \frac{\pi}{2\mu}, \quad (5)$$

and $u(\xi) = 0$, otherwise.

Here, λ, μ and β are parameters to be determined. The quantity μ represents the wave number, which is inversely proportional to the wavelength and characterizes the spatial frequency of the wave and c is the wave speed.

We compute the derivatives required for substitution.

For the sine-based ansatz (4):

$$\begin{aligned} u(\xi) &= \lambda \sin^\beta(\mu\xi), \\ u^n(\xi) &= \lambda^n \sin^{n\beta}(\mu\xi), \\ (u^n)_\xi &= n\mu\beta\lambda^n \cos(\mu\xi)\sin^{n\beta-1}(\mu\xi), \\ (u^n)_{\xi\xi} &= -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) \\ &\quad + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi). \end{aligned} \quad (6)$$

Similarly, for the cosine-based ansatz (5):

$$\begin{aligned} u(\xi) &= \lambda \cos^\beta(\mu\xi), \\ u^n(\xi) &= \lambda^n \cos^{n\beta}(\mu\xi), \\ (u^n)_\xi &= -n\mu\beta\lambda^n \sin(\mu\xi)\cos^{n\beta-1}(\mu\xi), \\ (u^n)_{\xi\xi} &= -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) \\ &\quad + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi). \end{aligned} \quad (7)$$

In the same way, we calculate the other necessary derivatives according to the order of the ordinary differential equation (ODE).

Finally, these expressions are substituted into equation (2), and the resulting terms involving trigonometric functions are balanced by comparing their powers. This yields a system of algebraic equations for the parameters λ, μ, and β, which can then be solved analytically.

In equations involving two spatial variables, the transformation becomes $\xi = x + y - ct$, with $u(x,y,t) = u(\xi)$, maintaining the method's applicability [11, 12, 13].

The *Sine-Cosine* method allows us to obtain solitary wave solutions for dispersive models with significantly reduced symbolic complexity. This type of exact solution is applicable to well-known models such as KdV, BBM, KP, and Boussinesq, all of which have constant coefficients.

The main advantage of this method is its broad applicability to a wide range of differential equations.

Another significant aspect of the method is its capability to substantially reduce the computational workload, enabling the use of symbolic computation tools for more complex calculations.

For our computations, we used Python [14] along with appropriate libraries such as SymPy [15] and NumPy [16], which facilitated both the analytical manipulation and the numerical evaluation of the solutions. Figure 1. summarizes the procedure as a flowchart.

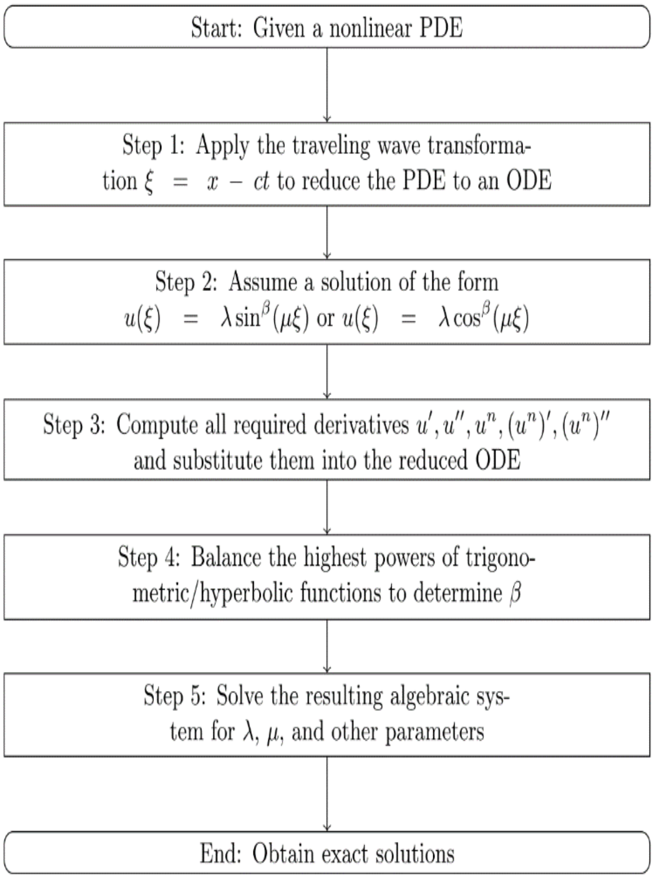


Figure 1. Flowchart of the Sine-Cosine method applied to nonlinear partial differential equations for deriving exact solitary wave solutions.

2. Application of the Sine-Cosine Method

This section presents the use of the Sine-Cosine method to construct exact solutions for nonlinear dispersive equations within the Camassa-Holm hierarchy.

2.1 Camassa-Holm hierarchy equations

This subsection introduces the general form of the Camassa-Holm-type equation used as the foundation of this study. By assigning specific values to its parameters, we recover four well-known nonlinear dispersive models: the Camassa-Holm (CH), Degasperis-Procesi (DP), Fornberg-Whitham (FW), and Fuchssteiner-Fokas-Camassa-Holm (FFCH) equations. These models are relevant in shallow water theory and are known for their rich mathematical structures, making them suitable candidates for the construction and classification of exact wave solutions. We begin with the generalized form:

$$u_t - u_{xxt} + au_x + buu_x = ku_xu_{xx} + uu_{xxx}, \quad (8)$$

where $u(x,t)$ denotes the wave profile, $a, b,$ and k are constants that characterize dispersion and nonlinearity. Specifically, a is related to the linear dispersion effects, b influences the nonlinear wave interactions, and k affects the strength of the nonlinearity in the wave evolution. The function $u(x,t)$ represents the fluid velocity or the free surface of the water. For $b = 3$ and $k = 1$, equation (8) takes the form:

$$u_t - u_{xxt} + au_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \quad (9)$$

This equation, proposed by Robert Camassa and Darryl Holm in 1993, serves as a nonlinear model used to describe the unidirectional propagation of nonlinear shallow-water waves over a flat bottom.

One of the key properties of the Camassa-Holm equation is its bi-Hamiltonian structure, meaning that it admits two distinct Hamiltonian formulations. This feature guarantees the existence of multiple conservation laws, which is a hallmark of integrable systems. The bi-Hamiltonian property plays a crucial role in understanding the integrability of the equation and allows for the application of powerful analytical techniques, such as the inverse scattering transform.

Another interesting aspect is that for $a = 0$, the equation admits a novel class of solitary waves with a discontinuous slope at the crest. These waves, known as peakons [17], possess a nonanalytic structure, differing from smooth solitons. Peakons exhibit discontinuities in their spatial derivative, where both one-sided spatial derivatives exist but differ only by a sign. Furthermore, for $a = 0$, the CH equation allows multi-soliton solutions composed of peaked solitary waves. However, these interesting solutions cannot be obtained with the method discussed in this work.

The equation has been the focus of numerous mathematical studies and finds applications in various fields.

We will study in this section four well-known equations of this family for the following specific values of the constants b and k. The set of constants

$$b = 3,\ k = 2,$$
$$b = 4,\ k = 3,$$
$$b = 1,\ k = 3,\ a = 1,$$
$$b = 3,\quad k = 2,\ a \text{ is replaced by } 2a,$$

which yield the following equations:

$$u_t - u_{xxt} + au_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (10)$$

$$u_t - u_{xxt} + au_x + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (11)$$

$$u_t - u_{xxt} + u_x + uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (12)$$

$$u_t - u_{xxt} + 2au_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \quad (13)$$

The Camassa-Holm (10) equation is part of a broader class of equations that includes the D, egasperis-Procesi

(11) equation [18], the Fornberg-Whitham (12) equation [19], and the Fuchssteiner-Fokas-Camassa-Holm (13) equation [20, 21]. These equations share both a linear dispersion term u_{xxt} and a nonlinear dispersion term uu_{xxx}.

2.2 Application to the Generalized Equation

In the following subsection, we apply the *Sine-Cosine* method directly to the generalized form of the Camassa-Holm hierarchy in order to derive an explicit solitary wave solution expressed in terms of its parameters.

Next, we make the substitution $u(x,t) = \lambda\cos^{\beta}(\mu\xi)$, where $\xi = x - ct$. With this substitution, the equation (8) transforms into

$$-cu' + cu''' + au' + \frac{b}{2}(u^2)' = \frac{k-1}{2}((u')^2)' + (uu'')'. \tag{14}$$

After integrating once, organizing terms, and canceling the integration constants, we obtain:

$$(a-c)u + cu'' + \frac{b}{2}u^2 = \frac{k-1}{2}(u')^2 + uu''. \tag{15}$$

Since we are seeking solutions in the form $u(\xi) = \lambda\cos^{\beta}(\mu\xi)$, the derivatives involved are determined accordingly,

$$\begin{aligned} u'(\xi) &= -\lambda\beta\mu\cos^{\beta-1}(\mu\xi)\sin(\mu\xi), \\ u''(\zeta) &= \lambda\beta(\beta-1)\mu^2\cos^{\beta-2}(\mu\xi) - \lambda\beta^2\mu^2\cos^{\beta}(\mu\xi). \end{aligned} \tag{16}$$

We substitute these expressions in (15) and obtain

$$\begin{aligned} &(a-c)\lambda\cos^{\beta}(\mu\xi) + c\lambda\beta(\beta-1)\mu^2\cos^{\beta-2}(\mu\xi) - c\lambda\beta^2\mu^2\cos^{\beta}(\mu\xi) + \frac{b}{2}\lambda^2\cos^{2\beta}(\mu\xi) - \frac{k-1}{2}\lambda^2\beta^2\mu^2\cos^{2\beta-2}(\mu\xi) \\ &+ \frac{k-1}{2}\lambda^2\beta^2\mu^2\cos^{2\beta}(\mu\xi) - \lambda^2\beta(\beta-1)\mu^2\cos^{2\beta-2}(\mu\xi) + \lambda^2\beta^2\mu^2\cos^{2\beta}(\mu\xi) = 0. \end{aligned} \tag{17}$$

To ensure that the equation remains consistent and does not contain terms with different powers of $\cos(\mu\xi)$, we must balance the exponents. This balance condition leads to the equations:

1. $2\beta = \beta - 2$, then $\beta = -2$.
2. $2\beta - 2 = \beta$, then $\beta = 2$.

These values guarantee that the terms in the resulting equation have compatible exponents, allowing us to solve the system of equations for λ, μ.

For $\beta = -2$, we obtain the following system of equations:

$$\begin{aligned} &(a-c)\lambda - 4c\lambda\mu^2 = 0, \\ &6c\lambda\mu^2 + \frac{b}{2}\lambda^2 + 2(k+1)\lambda^2\mu^2 = 0, \\ &-2(k+2)\lambda^2\mu^2 = 0. \end{aligned} \tag{18}$$

Solving this system, we find:

$$\begin{aligned} \mu &= \pm\sqrt{\frac{a-c}{4c}}, \\ \lambda &= \frac{-6c(a-c)}{bc + (k+1)(a-c)}, \\ \beta &= -2. \end{aligned} \tag{19}$$

From the previous calculations, it is concluded that the solution $u(x,t)$ takes the form:

$$u(x,t) = \left[\frac{-6c(a-c)}{bc+(k+1)(a-c)}\right] \times \cos^{-2}\left(\pm\sqrt{\frac{a-c}{4c}}(x-ct)\right). \tag{20}$$

On the other hand, in case $\beta = 2$, we obtain the system of equations:

$$\begin{aligned} &(a-c)\lambda - 4c\lambda\mu^2 - 2k\lambda^2\mu^2 = 0, \\ &\frac{b}{2}\lambda^2 + 2(k+1)\lambda^2\mu^2 = 0, \\ &2c\lambda\mu^2 = 0. \end{aligned} \tag{21}$$

Solving system (21), we obtain:

$$\mu = \pm\sqrt{\frac{-b}{4(k+1)}}, \tag{22}$$

$$\lambda = \frac{-2\big((a-c)(k+1)+bc\big)}{bk},$$

$$\beta = 2.$$

In this case, the solution $u(x,t)$ takes the form:

$$u(x,t) = \frac{-2\big((a-c)(k+1)+bc\big)}{bk} \times \cos^2\left(\pm\sqrt{\frac{-b}{4(k+1)}}(x-ct)\right). \tag{23}$$

To conclude, let us recall the identity that relates hyperbolic functions and trigonometric functions

$$\cos(ix) = \cosh(x). \tag{24}$$

2.3 Solutions for Particular Cases

In this subsection, we assign specific values to the parameters of the generalized Camassa-Holm-type equation in order to recover four well-known models: CH, DP, FW, and FFCH. By substituting these values into the general solution previously obtained, we derive explicit solitary wave solutions for each equation and express them in closed analytical form.

Using identity (24), we obtain the following set of solutions $u(x,t)$ for equations (10)-(13), respectively:

Case $\beta = -2$:

For CH equation (10):

$$u(x,t) = \frac{2c(c-a)}{a} \times \operatorname{sech}^2\left(\pm\sqrt{\frac{c-a}{4c}}(x-ct)\right), \tag{25}$$

where $a < c$.

For DP equation (11):

$$u(x,t) = \frac{3c(c-a)}{2a} \times \operatorname{sech}^2\left(\pm\sqrt{\frac{c-a}{4c}}(x-ct)\right), \tag{26}$$

where $a < c$.

For FW equation (12):

$$u(x,t) = \frac{6c(c-1)}{4-3c} \times \operatorname{sech}^2\left(\pm\sqrt{\frac{c-1}{4c}}(x-ct)\right), \tag{27}$$

where $1 < c$.

For FFCH equation (13):

$$u(x,t) = \frac{6c(c-2a)}{6a+c} \times \operatorname{sech}^2\left(\pm\sqrt{\frac{c-2a}{4c}}(x-ct)\right), \tag{28}$$

where $2a < c$.

Case $\beta = 2$:

For CH equation (10):

$$u(x,t) = -a\cosh^2\left(\pm\frac{1}{2}(x-ct)\right). \tag{29}$$

For DP equation (11):

$$u(x,t) = -\frac{2}{3}\cosh^2\left(\pm\frac{1}{2}(x-ct)\right). \tag{30}$$

For FW equation (12):

$$u(x,t) = \frac{6c-8}{3}\cosh^2\left(\pm\frac{1}{4}(x-ct)\right). \tag{31}$$

For FFCH equation (13):

$$u(x,t) = -2a\cosh^2\left(\pm\frac{1}{2}(x-ct)\right). \tag{32}$$

3.4 Classification and Discussion of Solitary Wave Profiles

The solutions obtained in the previous subsection can be classified into two distinct families of solitary wave profiles, depending on the exponent β used in the assumed functional form. When $\beta = -2$, the solutions are expressed in terms of the squared hyperbolic secant function (sech^2), while for, $\beta = 2$, they take the form of a squared hyperbolic cosine function $(\cosh^2)$.

The sech^2-type solutions correspond to localized waveforms that exhibit rapid decay at infinity and are typical of classical soliton profiles. These solutions represent smooth, symmetric solitary waves with a well-defined peak and exponential tails. In contrast, the $\cosh^2$-type solutions display a bell-shaped profile that grows away from the origin and represents a different class of solitary wave behavior. Although not strictly localized, these solutions are still smooth and symmetric but differ significantly in their spatial decay and amplitude modulation.

3. Visualization of Derived Solutions

In this section, we present graphical depictions of the exact solitary-wave solutions obtained for the Camassa–Holm hierarchy of equations.

The solutions fall into two categories: those with $\beta = -2$, which exhibit a hyperbolic secant squared profile, and those with $\beta = 2$, which have a squared hyperbolic cosine structure. The following figures compare these two solution types for each equation in the family.

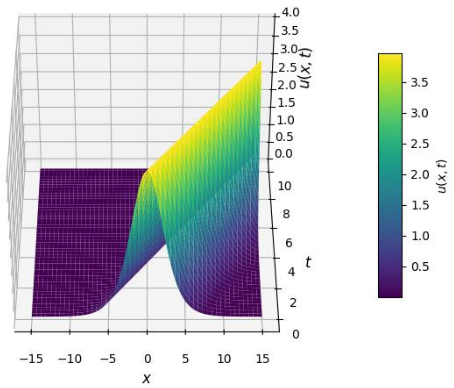


Figure 2: Solution of the Camassa-Holm equation for $\beta = -2$.

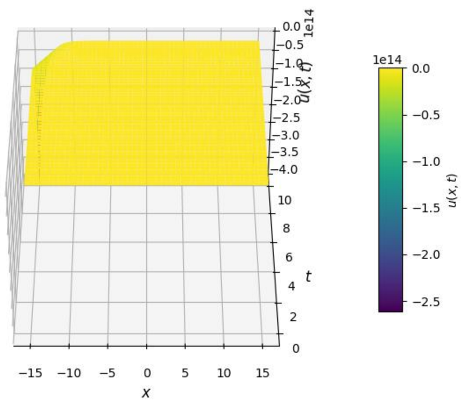


Figure 3: Solution of the Camassa-Holm equation for $\beta = 2$.

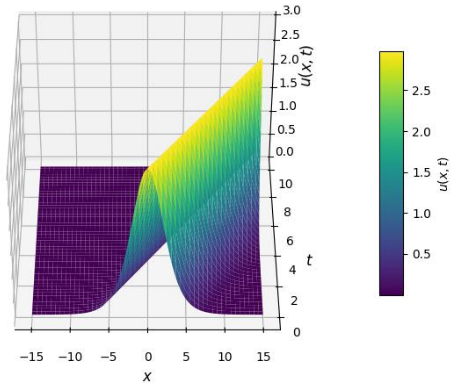


Figure 4: Solution of the Degasperis-Procesi equation for $\beta = -2$.

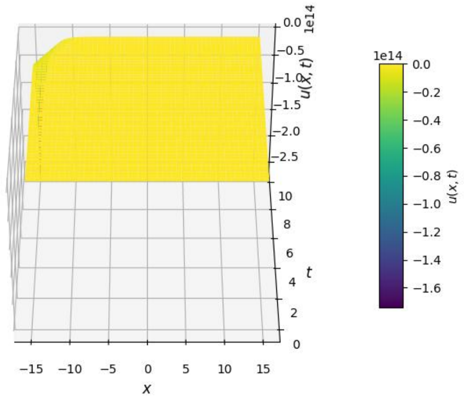


Figure 5: Solution of the Degasperis-Procesi equation for $\beta = 2$.

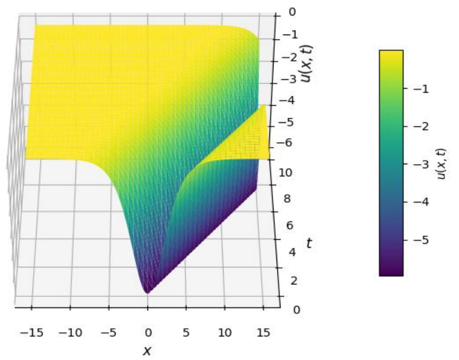


Figure 6: Solution of the Fornberg-Whitham equation for $\beta = -2$.

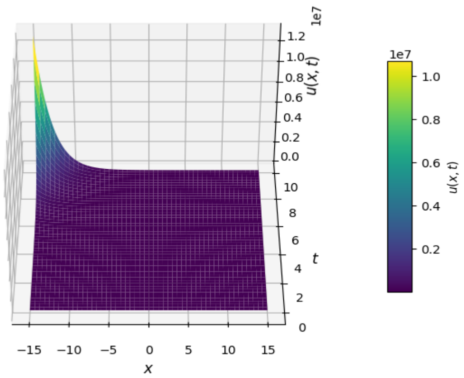
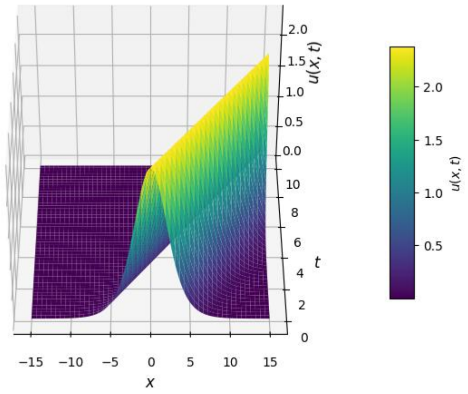


Figure 7: Solution of the Fornberg-Whitham equation for $\beta = 2$.



$\beta = 2.$

Figure 8: Solution of the Fuchssteiner-Fokas-Camassa-Holm equation for $\beta = -2.$

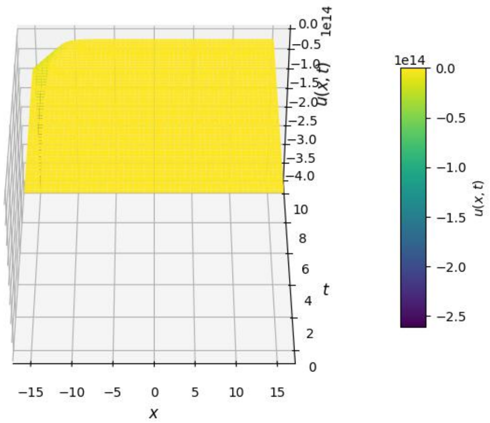


Figure 9: Solution of the Fuchssteiner-Fokas-Camassa-Holm equation for $\beta = 2.$

4. Conclusion

In this work, we derived exact solutions for a hierarchy of Camassa-Holm-type equations using the sine-cosine method. The obtained solutions are expressed in terms of elementary functions, including sine, cosine, hyperbolic sine, and hyperbolic cosine. These results demonstrate the method's effectiveness in constructing smooth solitary wave solutions for nonlinear dispersive equations. However, it is important to note that the method does not capture peaked solitary waves (peakons), which are a characteristic feature of the original Camassa-Holm equation.

Additionally, we explored the application of the sine-cosine method to other nonlinear models. Our experience suggests that when higher-order derivatives are involved -as in the case of the Ostrovsky equation- the method faces significant limitations. In such cases, it becomes extremely difficult, or even impossible, to achieve a consistent balance of exponents and coefficients after applying the traveling wave reduction and ansatz substitution.

Recent studies have shown that the sine-cosine method can be extended to systems of coupled equations and to equations with time-dependent coefficients. These extensions are promising but present new challenges. In particular, the presence of time-dependent terms complicates the analysis and restricts the domain of validity of the solutions. Future research could focus on refining the method for such generalized scenarios or on combining it with numerical approaches to address its current limitations.

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Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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