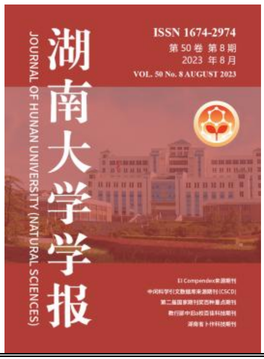

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Definability and Strength of Fourier Series Transform Obtained by Employing Parametric Integration

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Abstract: Regardless of the term coinage, this article is a story of adding variables that act in a similar way to the "s" in the Laplace transform. This idea is similar to the avoiding of points in improper integration, but instead of taking limits, the endpoints of each maximum integrable interval are made to be variables, but they still preserve the orders. The result is here called a parametric integral. It is applied here to Fourier series transform as an example. Therefore, some functions that formerly had no Fourier series expansion may no longer exist if one employs parametric integration. The resulting transform has advantages in terms of definability and strength. In definability because the transform exists for more functions than even for the well-known Fourier transform. It gives hope for solving more problems that may be of certain practical interests, especially when endpoint variables in the solution are replaced by respective limiting processes. The possibility that a solution still contains non-eliminable variables may lead to interesting non-standard analysis with the parameters as new "numbers", some of which are given here. The parameters may also serve in determining the partial inverses of the transform.

Keywords: Fourier series expansion, Fourier series transform, half-range expansion, Laplace transform, parametric integration.

采用参数积分获得的傅里叶级数变换的可定义性和强度

摘要：无论术语如何创造，本文都是一个添加变量的故事，这些变量的作用与拉普拉斯变换中的"s"类似。这个想法类似于避免不正确积分中的点，但不是取极限，而是将每个最大可积区间的端点设为变量，但它们仍然保留顺序。这里的结果称为参数积分。这里以傅里叶级数变换为例。因此，如果采用参数积分，一些以前没有傅里叶级数展开的函数可能不再存在。由此产生的变换在可定义性和强度方面具有优势。可定义性是因为该变换存在的函数比众所周知的傅里叶变换还要多。它为解决更多可能具有一定实际意义的问题带来了希望，特别是当解决方案中的端点变量被各自的限制过程替换时。解决方案仍然包含不可消除变量的可能性可能会导致有趣的非标准分析，其中参数作为新的"数字"，其中一些在此处给出。这些参数还可以用于确定变换的部分逆。

关键词：傅里叶级数展开、傅里叶级数变换、半程展开、拉普拉斯变换、参数积分。

1. Introduction

Consider the integral $\int_{\pi/2}^{\pi} \tan x\, dx$. Then, it is clearly a divergent integral, i.e., its value as a real number does not exist. On the other hand, the collection of values of $\int_{\pi/2}^{t} \tan x\, dx$ where $\frac{\pi}{2} < t < \pi$ does somehow represent the above divergent integral. In this paper, we shall pursue this line of thinking in a more general manner and apply it to handle transforms.

The main idea of using parameters here is to assign values other than numerical to calculations that formerly failed to give numerical values in certain situations. For example, the Riemann integral of a function $f: I \to R$ over an interval I is defined as the supremum as well as the infimum of the sums $\sum_{n=0}^{N} f(x_n)\Delta x_n$ where $\Delta x_n = x_{n+1} - x_n,\ x_n < x_{n+1}$ for each n, in which $I = (x_0, x_{N+1})$, *if the value exists*. Instead of giving up on the clause "if the value exists", one might say that the Riemann integral of f over I is the sum $\sum_{n=0}^{N} f(x_n)\Delta x_n$ where both N and $x_1, \cdots, x_N$ are "parameters", namely N is a positive integer valued variable, while $x_1, \cdots, x_N$ are real valued variables with the above properties. This differs from conceptual variations of integration, such as in [1–3], in that it does not propose conceptual improvements but expands the definability for Riemann integration. However, in some cases, this approach might be of practical interest, for example, when the output with the parameters does not serve as a destination but as a transitional entity to be further processed, such as in the case of transforms.

Instead of employing an improper integration concept such as in [4], this article deals with the following. Suppose the *parameter-introducing operator* is an operator $P: A \to B$, where A is the set of all unions of the form $\bigcup_{n=-\infty}^{\infty} I_n$ of intervals in the real line R which are partitions of $R - X$ for X the set of all endpoints of the I_n's, while B is the set of all J-restricted unions $J \cap \bigcup_{n=-\infty}^{\infty} I_n^*$ where I_n^* is the interval $(R_a, L_b) \stackrel{\text{def}}{=} \{x | R_a < x < L_b\}$ for parameters R_a and L_b if $I_n = (a, b)$ and $J = (R_{-\infty}, L_{\infty})$. The points a and b are allowed to be $-\infty$ and ∞ respectively. Thus, P sends the union $S = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$ to the union $P(S) = J \cap \left((R_{-\infty}, L_{-3}) \cup (R_{-3}, L_2) \cup (R_2, L_{\infty})\right)$. While P sends $T = \bigcup_{n=-\infty}^{\infty} (n, n+1)$ to $P(T) = J \cap \bigcup_{n=-\infty}^{\infty} (R_n, L_{n+1})$. The operator P induces a transformation that sends any formal integral of any function symbol f over the sets in A to its corresponding formal integral over the sets in B. Thus, the formal integral

$$\int_S f(x)dx = \int_{-\infty}^{-3} f(x)dx + \int_{-3}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx \tag{1}$$

is sent via P to

$$\int_{P(S)} f(x)dx = \int_{J\cap(R_{-\infty}, L_{-3})} f(x)dx + \int_{J\cap(R_{-3}, L_2)} f(x)dx + \int_{J\cap(R_2, L_{\infty})} f(x)dx \tag{2}$$

Thus, $\int_{P(T)} f(x)dx$ is just the formal sum $\sum_{R_{-\infty} \le R_n \& L_{n+1} \le L_{\infty}} \int_{R_n}^{L_{n+1}} f(x)dx$. The integral of the form $\int_{P(T)} f(x)dx$ is called the parametric integral of f over R with constraint T.

Returning to our very first integral example, an illustration of a concrete case of the above discussion might be appropriate.

$$\int_{-2\pi}^{3\pi/2} \to \tan x\, dx = \int_{-2\pi}^{L_{-\frac{3\pi}{2}+\frac{n\pi}{2}}} \tan x\, dx + \sum_{n=0}^{6} \int_{R_{-\frac{3\pi}{2}+\frac{n\pi}{2}}}^{L_{-\frac{3\pi}{2}+\frac{n\pi}{2}+\frac{\pi}{2}}} \tan x\, dx = -\ln\left(\cos L_{-\frac{3\pi}{2}+\frac{n\pi}{2}} \prod_{n=0}^{6} \frac{\cos L_{-\frac{3\pi}{2}+\frac{n\pi}{2}+\frac{\pi}{2}}}{\cos R_{-\frac{3\pi}{2}+\frac{n\pi}{2}}}\right) \tag{3}$$

where the point 2π is not replaced by a point parameter because the integration has no problem there.

An unbounded version of the above is the integral:

$$\int_{-\infty}^{\infty} \to \tan x\, dx = \int_{-L}^{L_{-\left\lfloor\frac{L}{\pi}\right\rfloor\frac{\pi}{2}}} \tan x\, dx + \sum_{n=-\left\lfloor\frac{L}{\pi}\right\rfloor}^{\left\lfloor\frac{L}{\pi}\right\rfloor} \int_{R_{\frac{n\pi}{2}}}^{L_{\frac{n\pi}{2}+\frac{\pi}{2}}} \tan x\, dx + \int_{R_{\left\lfloor\frac{L}{\pi}\right\rfloor\frac{\pi}{2}}}^{L} \tan x\, dx =$$
$$-\ln\left(\left(\frac{\cos L_{-\left\lfloor\frac{L}{\pi}\right\rfloor\frac{\pi}{2}}}{\cos L_{\left\lfloor\frac{L}{\pi}\right\rfloor\frac{\pi}{2}}}\right)\left(\prod_{n=-\left\lfloor\frac{L}{\pi}\right\rfloor}^{\left\lfloor\frac{L}{\pi}\right\rfloor} \frac{\cos L_{\frac{n\pi}{2}+\frac{\pi}{2}}}{\cos R_{\frac{n\pi}{2}}}\right)\right) \tag{4}$$

One may formally apply the variable integral concept to any discussion involving any type of integration such as the gauge integral introduced in [5], [6] or McShane integral in [7] or even the integral symbol might be interpreted as a discrete sum or a very different concept altogether, but in this article, the Fourier series transform is discussed instead. One of the motivations of adopting this choice is that the Fourier series is seemingly more restrictive in the kind of functions required to make it work, besides the sheer curiosity of what the Fourier series spectra look like of more arbitrary functions than the more familiar ones, albeit the former might be in the forms that contain parameters. In fact, some of the parameters might be eliminable when after some manipulations, they are replaced by the operations "$\lim_{a<R_a\to a}\cdots$" and "$\lim_{b>L_b\to b}\cdots$" for each R_a and L_b respectively any time the limits exist, and leaving them alone when they do not.

1.1. Basic Theory

The Fourier series presented here is the classic one instead of the more derived forms, such as in [8-10] or [11], for the sake of emphasizing the presentation of the effects of the treatments described in the introduction to the theory. The following are standard definitions and results for Fourier series.

1.1.1. Definition 1

(Complex) Fourier Series [12]: A Fourier series *is a series of the forms*

$$f(x) = \sum_{n=1}^{\infty} C_n \exp i\frac{n\pi x}{L} \quad (5)$$

Theorem 1. Euler Identities [12]: The coefficients of the Fourier series satisfy the following equations:

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-i\frac{n\pi x}{L}\right)dx \quad (6)$$

1.1.2. Definition 2

A function f is said to be expandable into a Fourier series with period $2\mathrm{L} > 0$ if there are constants $\cdots \mathrm{C}_{-1}, \mathrm{C}_0, \mathrm{C}_{3,\dots}$ such that

$$f(x) = \sum_{n=1}^{\infty} C_n \exp i\frac{n\pi x}{L} \quad (7)$$

for all x except at a set of zero measures [12].

Oddly enough, as discovered in [13], periodicity is not an imperative condition for the existence of Fourier series.

Theorem 2. Representation by the Fourier Series [12]: Suppose f(x) has the properties that:

1. It is piecewise continuous,
2. It has positive period $2\mathrm{L} > 0$,
3. It has left-hand and right-hand side derivatives at any point except at countably many points in the interval $[-\mathrm{L}, \mathrm{L}]$, then f is expandable into a Fourier series with period 2L and the value of the series at a discontinuous point x is $\frac{1}{2}\left(\mathrm{f}(\mathrm{x}-) + \mathrm{f}(\mathrm{x}+)\right)$ where $\mathrm{f}(\mathrm{x}-)$ and $\mathrm{f}(\mathrm{x}+)$ are the left-hand and right-hand limits of f at x.

2. Research Method

The path to be pursued here is of the "global" analysis of the Fourier series as opposed to local investigations as in [14]. In addition, unlike [15] which delves through considerable abstraction, here a rather concrete approach is pursued.

The main idea behind the following definition is to isolate undesirable points at distances from the rest of the set considered. Topologically speaking [16], the real line R as a topological space is stripped from open sets, one for each undesirable point that causes unboundedness of a certain functional, but the open sets are variables, each from the collection of all open sets containing the failure point. When the open set variables are replaced by any of their possible instances, the functional becomes bounded [17].

Let $f: R \to R$ be a function, $\{a_n\}_{n=-\infty}^{\infty}$ be an integer-indexed sequence in R with $a_m \le a_n$ if $m < n$, and $S = \bigcup_{n=-\infty}^{\infty}(a_n, a_{n+1})$. Then, as already mentioned in the Introduction, the parametric integral of f over R with constraint S is the formal integral:

$$\int_S f(x)dx = \sum_{R_{-\infty}\le R_{a_n} \& L_{a_{n+1}}\le L_\infty} \int_{R_{a_n}}^{L_{a_{n+1}}} f(x)dx \quad (8)$$

When replacements with parameters are only done to failure points among the a_n's, $-\infty$, or ∞, namely points that make the integral fail to exist, it is denoted by $\int_a^b \to f(x)dx$.

Now comes the apex of this paper, which establishes the validity of its research methodology to more definability, even that of the Fourier integral [12].

Theorem 3. Existence of Fourier Series under Parametric Integration: Let $\mathrm{f}: \mathrm{R} \to \mathrm{R}$ be a function integrable for each finite interval except maybe at points a_n for integers n and maybe on some infinite intervals. Then, the parametric integration gives the Fourier series expansion:

$$f(x) = \sum_{n=1}^{\infty} C_n \exp i\frac{n\pi x}{L} \quad (9)$$

where the C_n's satisfies the Euler identities:

$$C_n = \frac{1}{2L}\int_{-L}^{L} \to f(x)\exp\left(-i\frac{n\pi x}{L}\right)dx \quad (10)$$

Proof: Suppose that $\int_a^b f(x)dx$ exists for any finite interval $[a, b]$, but $\int_{-\infty}^{\infty} f(x)dx$ does not exist. Let $R_{-\infty} < 0 < L_\infty$ and $L = \max\{|R_{-\infty}|, |L_\infty|\}$. Then:

$$f_L = \begin{cases} f(x), & \text{if } x \in (R_{-\infty}, L_\infty), \\ 0, & \text{if } x \in (-L, L) - (R_{-\infty}, L_\infty) \end{cases} \quad (11)$$

Then $C_n(L) = \frac{1}{2L}\int_{-L}^{L} f_L(x)\exp\left(-i\frac{n\pi x}{L}\right)dx$ exists for any integer n, thus also the expansion $f_{R_{-\infty},L_\infty}(x) = \sum_{n=1}^{\infty} C_n(L)\exp\left(i\frac{n\pi x}{L}\right)$ for any $x \in (R_{-\infty}, L_\infty)$. Thus, the theorem is true for this type of f.

Now suppose $\int_u^v f(x)dx$ exists for any finite interval $[u, v]$, but $\int_u^v f(x)dx$ does not exist if $a \in (u, v)$. Let $L_a < a < R_a$. Suppose

$$f_{L_a,R_a} = \begin{cases} f(x), & \text{if } x \in R - (L_a, R_a), \\ 0, & \text{if } x \in (L_a, R_a). \end{cases} \quad (12)$$

According to the last paragraph, the function $\left(f_{L_a,R_a}\right)_{R_{-\infty},L_\infty}$ satisfies the theorem for any $x \in (R_{-\infty}, L_\infty) - (L_a, R_a)$ if the latter set is not empty.

So far, the Fourier Series Transform is not found by the authors in any article. Maybe because it is too specialized to be employed as transform, but here it is chosen because it might seem rather striking that periodicity is altogether removable from the list of requirements that a function be expandable as a Fourier series when parametric integration is used. In this article, some of its development mimicking the development of the Fourier integral is presented.

Suppose $f(x)$ is a function satisfying the existence conditions. Then, the *Fourier series transform of* $f(x)$ is the sequence $\{C_n\}_{n=-\infty}^{\infty}$ where $f(x)=\sum_{n=0}^{\infty} C_n \, exp\, i\frac{n\pi x}{L}$ for a positive number L. This is written simply as $(C_n)=F(f)$.

Theorem 4. Linearity: Suppose $\mathrm{F(f)=(C_n)}$ and $\mathrm{F(g)=(D_n)}$ and let α,β be real numbers. Then

$\mathrm{F(\alpha f+\beta g)=\alpha F(f)+\beta F(g)}$ (13)

Namely that $\mathrm{F(\alpha f+\beta g)=\alpha(C_n)+\beta(D_n)=(\alpha C_n+\beta D_n)}$.

Proof: Since $f(x)=\sum_{n=-\infty}^{\infty} C_n \exp i\frac{n\pi x}{L}$ and $g(x)=\sum_{n=-\infty}^{\infty} D_n \exp i\frac{n\pi x}{L}$, then:

$$\alpha f(x)+\beta g(x)=\alpha\sum_{n=-\infty}^{\infty} C_n \exp i\frac{n\pi x}{L}+\beta\sum_{n=-\infty}^{\infty} D_n \exp i\frac{n\pi x}{L}=\sum_{n=-\infty}^{\infty}(\alpha C_n+\beta D_n)\exp i\frac{n\pi x}{L} \quad (14)$$

Theorem 5: Transform of the derivative. $F(f')=\frac{n\pi}{L}(C_n)$ provided $F(f)=(C_n)$.

Proof: Since $f(x)=\sum_{n=-\infty}^{\infty} C_n \exp i\frac{n\pi x}{L}$ then $f'(x)=\sum_{n=-\infty}^{\infty}\frac{n\pi}{L} C_n \exp i\frac{n\pi x}{L}$.

Theorem 6: Transform of integral. $F\left(\int_0^x f(x)dx\right)=\frac{L}{n\pi}(C_n)$ provided $F(f)=(C_n)$.

Proof: It is well known that $f(x)=\left(\int_0^x f(t)dt\right)'$. Suppose $(C_n)=F(f)$ and $(C_n^*)=F\left(\int_0^x y(t)dt\right)$. According to Theorem 5, $F\left(\left(\int_0^x y(t)dt\right)'\right)=\frac{n\pi}{L}(C_n^*)$ and this should be equal to (C_n). Then for all n, $C_n=\frac{n\pi}{L}C_n^*$. Then for any n, $C_n^*=\frac{L}{n\pi}C_n$.

Theorem 7. Shifting of domain: If x is shifted to $x-a$, the transformation is

$F\big(f(x-a)\big)=\exp\left(-i\frac{n\pi a}{L}\right)F(f)$ (15)

Proof: Direct calculation yields that:

$$F\big(f(x-a)\big)=\left(\int_{-L}^{L} f(x-a)\ \exp\left(-i\frac{n\pi x}{L}\right)dx\right)=\int_{-L-a}^{L-a} f(x-a)\exp\left(-i\frac{n\pi(x-a)}{L}\right)\exp\left(-i\frac{n\pi a}{L}\right)d(x-a)=\int_{-L}^{L} f(x)\exp\left(-i\frac{n\pi x}{L}\right)\exp\left(-i\frac{n\pi a}{L}\right)dx=\exp\left(-i\frac{n\pi a}{L}\right)F(f) \quad (16)$$

Theorem 8. Difference in a transform: Let $\mathrm{F(f)=(C_n)}$. Then

$\mathrm{F^{-1}(\Delta C_n)=f(x)\left(\exp\left(-i\frac{\pi x}{L}\right)-1\right)}$ (17)

where $\Delta C_n=C_{n+1}-C_n$.

Proof: Calculating directly:

$$\sum_{n=-\infty}^{\infty}\Delta C_n \exp i\frac{n\pi x}{L}=\sum_{n=-\infty}^{\infty}(C_{n+1}-C_n)\exp i\frac{n\pi x}{L}=\sum_{n=-\infty}^{\infty}C_{n+1}\exp i\frac{n\pi x}{L}-\sum_{n=-\infty}^{\infty}C_n\exp i\frac{n\pi x}{L}=\sum_{n=-\infty}^{\infty}C_{n+1}\exp i\left(\frac{(n+1)\pi x}{L}-\frac{\pi x}{L}\right)-\sum_{n=-\infty}^{\infty}C_n\exp i\frac{n\pi x}{L}=\sum_{n=-\infty}^{\infty}C_{n+1}\ \exp i\frac{(n+1)\pi x}{L}\exp\left(-i\frac{\pi x}{L}\right)-\sum_{n=-\infty}^{\infty}A_n\cos\frac{n\pi x}{L}=\exp\left(-i\frac{\pi x}{L}\right)\sum_{n=-\infty}^{\infty}C_{n+1}\ \exp i\frac{(n+1)\pi x}{L}-\sum_{n=-\infty}^{\infty}C_n\cos\frac{n\pi x}{L}=f(x)\exp\left(-i\frac{\pi x}{L}\right)-f(x)=f(x)\left(\exp\left(-i\frac{\pi x}{L}\right)-1\right) \quad (18)$$

Theorem 9. Cumulative sum of the transform: Suppose $\mathrm{F(f)=(C_n)}$. Then

$\mathrm{F^{-1}\left(\sum_{k=0}^{n}C_k\right)=\frac{f(x)}{\exp\left(-i\frac{\pi x}{L}\right)-1}}$ (19)

where $\sum_{k=0}^{n}C_k=-\sum_{k=-n}^{0}C_k$ if $m=-n\le 0$.

Proof: Suppose

$\mathrm{(D_n)=\{\sum_{k=0}^{n}C_k\}_{n\in Z}=F(g)}$ (20)

Then $C_n=\Delta D_n=D_n-D_{n-1}$. According to Theorem 8,

$\mathrm{f(x)=g(x)\left(\exp\left(-i\frac{\pi x}{L}\right)-1\right)}$ (21)

Thus

$\mathrm{g(x)=\frac{f(x)}{\exp\left(-i\frac{\pi x}{L}\right)}}$ (22)

Given $\mathrm{F(f)=(C_n)}$ and $\mathrm{F(g)=(D_n)}$. The convolution of f and g is the function $f*g$ such that $\mathrm{F(f*g)=F(f)F(g)=(C_nD_n)}$.

Theorem 10. Explicit formula of convolution: The convolution of two functions f and g is expressed as

$\mathrm{(f*g)(x)=\int_{-L}^{L}f(\tau)g(x-\tau)\,d\tau}$ (23)

Proof: Suppose $\mathrm{F(f)=(C_n)}$ and $\mathrm{F(g)=(D_n)}$. By direct calculation,

$\mathrm{\left(\exp\left(i\frac{n\pi\tau}{L}\right)D_n\right)=F\big(g(x-\tau)\big)=\Big(\frac{1}{2L}\int_{-L}^{L}g(x-}$ τexpinπxLdx (24)

Then,

$$\mathrm{C_nD_n=\left(\int_{-L}^{L}f(\tau)\exp\left(i\frac{n\pi\tau}{L}\right)d\tau\right)D_n=\left(\int_{-L}^{L}f(\tau)\exp\left(i\frac{n\pi\tau}{L}\right)D_n d\tau\right)=\int_{-L}^{L}f(\tau)\left(\int_{-L}^{L}g(x-\tau)\exp\left(i\frac{n\pi x}{L}\right)dx\right)d\tau=\int_{-L}^{L}\left(\int_{-L}^{L}f(\tau)g(x-\tau)\,d\tau\right)\exp\left(i\frac{n\pi x}{L}\right)dx} \quad (25)$$

Hence,

$$(f * g)(x) = \int_{-L}^{L} f(\tau)g(x-\tau)\,d\tau. \ \square \qquad (26)$$

Theorem 11. Half-range expansion of a function $f(x)$ on the interval $[-L, L]$: Let $f: [-L, L] \to R$ be a function satisfying the existence conditions except for periodicity. Then $f(x) = F^{-1}(C_n)$ with $C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-i\frac{n\pi x}{L}\right)dx$ is an extension of f to (almost) the entire set R.

Proof: Suppose f(x) is extended to the entire real line R as a function f^* with period 2L. According to Theorem 2, $F(f^*) = (C_n)$ where

$$C_n = \frac{1}{2L}\int_{-L}^{L} f^*(x)\exp\left(-i\frac{n\pi x}{L}\right)dx = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-i\frac{n\pi x}{L}\right)dx \qquad (27)$$

Theorem 12. Half-range expansion for a function on an arbitrary interval: Let f(x) be a function satisfying the existence conditions apart from periodicity on the interval $J = [a, b]$. Then, on the interval J the following expression holds:

$$F(f) = \exp\left(i\frac{n\pi(a+L)}{L}\right)F(f(x+a+L)) \qquad (28)$$

where $L = \frac{b-a}{2}$.

Proof: Let $f^*(x) = f(x+a+L)$ on the interval $[-L, L]$ with $L = \frac{b-a}{2}$. According to Theorem 11, $F(f^*) = \left(\exp\left(-i\frac{n\pi(-a-L)}{L}\right)\right)F(f)$. Thus, $F(f) = \exp\left(i\frac{n\pi(a+L)}{L}\right)F(f^*)$ as desired.

3. Results and Discussion

Now, the discussion of the application of parametric integration to Fourier series transform is complete. First, one should define how to polish expressions to minimize the number of parameters, which is reminiscent of taking limits when improper integral exists.

Observing the improper integral definition as in [4], this concept is the best approximation. Let X be an expression containing some parameters. The limit of X is the expression obtained by taking the existing limits $\lim_{a<R_a\to a} X$ for all parameters of the form R_a and $\lim_{b>L_b\to b} X$ for all parameters of the form L_b, and leaving other R_a's and L_b's as they are. The limit of X is written as lim (X).

Now comes the main concept of this paper, namely a transform stronger than the Fourier transform. Let f(x) be a function satisfying the existence conditions apart from periodicity on R. Then, the following expression is called the parametric Fourier Series Transform of f:

$$F(\to f) :\overset{\text{def}}{=} (C_n) \qquad (26)$$

where the C_ns are calculated by parametric integration. According to Theorem 3, $F(\to f)$ exists and its inverse is

$$((C_n)) = \lim\left(\sum_{n=0}^{\infty} C_n \exp i\frac{n\pi x}{L}\right) \qquad (27)$$

For functions not defined in the whole R, the following can be used as its extension to the whole R. This is reminiscent of the half-range expansion. Let $D \subseteq R$ and $f: D \to R$ be a function. Let f^* be the function

$$f^*(x) = \begin{cases} 0, \text{if } x \notin D, \\ f(x), \text{if } x \in D \end{cases} \qquad (28)$$

Suppose for any $L > 0$, f^* restricted to $[-L, L]$ always has its parametric integral. Then, the *parametric Fourier Series Transform of f* is defined as $F(f) \overset{\text{def}}{=} F(\to f^*)$.

An example of the use of the transform is the solution of the differential equation $y'' + y' = \tan x$. We apply the transform to both sides to obtain the solution

$$E = \frac{(C_n)}{\left(\frac{n\pi}{L}\right)^2 + \frac{n\pi}{L}} \qquad (29)$$

where $F(\tan x) = (C_n)$, namely,

$$C_n = \frac{1}{2L}\int_{-L}^{L} \to \tan x \exp\left(-i\frac{n\pi x}{L}\right)dx \qquad (30)$$

A corollary is worth mentioning for such a solution E.

Corollary 1: Let f be a solution of an equation E. Then lim(f) is also a solution of E at all non-failure points.

Proof: If f contains no parameter then the corollary is trivial. Suppose f contains a parameter of the form R_a whose limit is taken in lim(f), namely R_a is in f but not in lim(f). Hence, $\lim_{a<R_a\to a} f$ exists for all x. Thus if E is true for say $R_a = r$ then E is true for any number s such that $a < s < r$. So E should be true for lim(f). A similar argument is used to prove the corollary for any other types of parameters.

Solutions of problems that retain non-eliminable parameters might lead to interpretations that they contain some "non-standard numbers" represented by the parameters, namely the L_a's and R_b's, the arithmetic of which can be developed further. The parameters L_a and R_b for real numbers a, b might be redefined as $\{(u, a) | u < a\}$ and $\{(b, v) | b < v\}$ respectively, while $R_{-\infty}$ and L_∞ as $\{(-\infty, v) | v \in R\}$ and $\{(u, \infty) | u \in R\}$ respectively. Any property P is attributed to a parameter among $L_a, R_b, R_{-\infty}$, or L_∞ if it holds for all $x \in I$ where I is a member of the said parameter. Thus, one says for example that $P(L_a)$ if and only if there is an $I \in L_a$ such that for all $x \in I$ the statement P(x) holds. In proofs, it is convenient to use the clause "by locality of x, $P(L_a)$ is true" when expressing the latter condition. An abuse of notation is conveniently used, namely $x \in v$ to denote that x is a possible value of the variable v. Applied to the parameters, of course "$x \in L_a$" means "$(\exists I \in L_a)(x \in I$".

The following theorem states some rules for the arithmetic of parameters.

Theorem 13: If a and b are real numbers, unless specified, then:

a) $-L_a = R_{-a}$ and $-R_a = L_{-a}$. Also: $-L_\infty = R_{-\infty}$ and $-R_{-\infty} = L_\infty$.

b) $\frac{1}{R_0} = L_\infty$ and $\frac{1}{L_0} = R_{-\infty}$. Also, $\frac{1}{R_{-\infty}} = L_0$ and $\frac{1}{L_\infty} = R_0$.

c) If $b \neq 0$, $\frac{1}{L_b} = R_{\frac{1}{b}}$ and $\frac{1}{R_b} = L_{\frac{1}{b}}$.

d) $a + R_b = R_a + R_b = R_{a+b}$ and $a + L_b = L_a + L_b = L_{a+b}$.

e) $a - L_b = R_a - L_b = R_{a-b}$ and $a - R_b = L_a - R_b = L_{a-b}$.

f) If $a, b > 0$ then $aR_b = R_aR_b = R_{ab}$ and $aL_b = L_aL_b = L_{ab}$.

g) If $a, b < 0$ then $aR_b = R_aR_b = L_{ab}$ and $aL_b = L_aL_b = R_{ab}$.

h) If $a < 0$ but $b > 0$ then $aR_b = L_aR_b = L_{ab}$ and $aL_b = R_aL_b = R_{ab}$.

i) If $a > 0$ but $b < 0$ then $aR_b = R_{ab}$ and $L_aR_b = L_{-ab}$. Also $aL_b = L_{ab}$ and $R_aL_b = R_{-ab}$.

j) If $a, b > 0$, $\frac{a}{R_b} = L_{\frac{a}{b}}$ and $\frac{R_a}{R_b} = L_{\frac{a}{b}}$. Also, $\frac{a}{L_b} = L_{\frac{a}{b}}$ and $\frac{L_a}{L_b} = R_{\frac{a}{b}}$.

k) If $a, b < 0$, $\frac{a}{R_b} = L_{\frac{a}{b}}$ and $\frac{R_a}{R_b} = L_{\frac{a}{b}}$. Also, $\frac{a}{L_b} = R_{\frac{a}{b}}$ and $\frac{L_a}{L_b} = R_{\frac{a}{b}}$.

l) If $a < 0$, $b > 0$, $\frac{a}{R_b} \leq L_{\frac{a}{b}} \leq \frac{a}{L_b}$ and $\frac{R_a}{R_b} \leq R_{ab} \leq \frac{L_a}{R_b}$. Also, $\frac{a}{L_b} \leq R_{\frac{a}{b}} \leq \frac{a}{R_b}$ and $\frac{L_a}{L_b} \leq L_{ab} \leq \frac{R_a}{L_b}$.

m) If $a > 0, b < 0$, $L_{\frac{a}{b}} = \frac{a}{R_b}$ and $\frac{R_a}{R_b} \leq R_{\frac{a}{b}} \leq \frac{R_a}{L_b}$. Also, $R_{\frac{a}{b}} = \frac{a}{L_b}$ and $\frac{L_a}{L_b} \leq L_{\frac{a}{b}} \leq \frac{L_a}{R_b}$.

Proof: Because the proofs are rather routine, we shall mention only some examples of them, namely for parts a, b, e, f, and m only.

a) Let $x \in L_a$. Then $x < a$. Hence $-a < -x$, namely $-x \in R_{-a}$. By arbitrariness of x, $-L_a \leq R_{-a}$. Conversely, if $x \in R_{-a}$ then $-a < x$. Hence, $-x < a$, namely $-x \in L_a$. Hence $x \in -L_a$. Arbitrariness of x implies $R_{-a} \leq -L_a$. Thus $-L_a = R_{-a}$.

The $-R_a = L_{-a}$ part is proven similarly.

Suppose $x \in L_\infty$. Then $x < \infty$. Hence, $-\infty < -x$. Therefore, $-x \in R_{-\infty}$.

Arbitrariness of x yields $-L_\infty \leq R_{-\infty}$.

Conversely suppose $x \in R_{-\infty}$. Then $-\infty < x$. Hence $-x < \infty$. So $-x \in L_\infty$. Thus, $x \in -L_\infty$. Arbitrariness of x yields.

$R_{-\infty} \leq -L_\infty$, establishing $-L_\infty = R_{-\infty}$.

The $-R_{-\infty} = L_\infty$ part is proven similarly.

Suppose $x \in R_0$. Then $0 < x$. Hence, $\frac{1}{x} < \infty$, namely $\frac{1}{x} \in L_\infty$. Arbitrariness of x yields $\frac{1}{R_0} \leq L_\infty$. Conversely, if $x \in L_\infty$. Locally, $x > 0$. Thus $0 < \frac{1}{x}$. Hence $\frac{1}{x} \in R_0$, namely $x \in \frac{1}{R_0}$. arbitrariness of x makes $L_\infty \leq \frac{1}{R_0}$, establishing $L_\infty = \frac{1}{R_0}$.

The $\frac{1}{L_0} = R_{-\infty}$ part is proven similarly, and so are the remaining cases.

e) From Parts a and d, $a - L_b = a + R_{-b} = R_{a-b}$. Also $R_a - L_b = R_a + R_{-b} = R_{a-b}$.

The rest is proven similarly.

f) Suppose $a, b > 0$. Suppose $x \in R_b$. Then, $b < x$. Hence $ab < ax$, namely $ax \in R_{ab}$. Since x is arbitrary, $aR_b \leq R_{ab}$. Consequently, $\frac{1}{a}R_{ab} \leq R_{\frac{1}{a}ab} = R_b$. Therefore $R_{ab} \leq aR_b$ also, establishing $R_{ab} = aR_b$.

Now, if $x \in R_a$ and $y \in R_b$, then $a < x$ and $b < y$. Multiplying both inequalities, $ab < xy$. Hence $xy \in R_{ab}$. By arbitrariness of x and y, $R_aR_b \leq R_{ab}$. Conversely suppose $z \in R_{ab}$, then $ab < z$. Because $ab > 0$, then $z > 0$ also. Suppose $z = kab$, then $k > 1$ so that $\sqrt{k} > 1$. Thus $\sqrt{k}a \in R_a$. And $\sqrt{k}bR_b$. Thus $z \in R_aR_b$. Since z is arbitrary, $R_{ab} \leq R_aR_b$, establishing $R_{ab} = R_aR_b$.

The rest is proven similarly.

m) Suppose $a > 0, b < 0$. Let $x \in R_b$. Then $b < x$. Locality of x yields $b < x < 0$.

Because $a > 0$, $\frac{a}{x} < \frac{a}{b}$, then $\frac{a}{x} \in L_{\frac{a}{b}}$. Since x is locally arbitrary $\frac{a}{R_b} \leq L_{\frac{a}{b}}$. Conversely let $y \in L_{\frac{a}{b}}$. Then $y < \frac{a}{b}$. Hence $y < \frac{a}{b} < 0$. Thus, $b < \frac{a}{y}$, namely $\frac{a}{y} \in R_b$. Thus $y \in \frac{a}{R_b}$. Since y is arbitrary, $L_{\frac{a}{b}} \leq \frac{a}{R_b}$, establishing $L_{\frac{a}{b}} = \frac{a}{R_b}$.

Now suppose $x \in R_a$ and $y \in R_b$. Then, $a < x$ and $0 < b < y$. Thus $\frac{a}{b} < \frac{a}{y} < \frac{x}{y}$, namely $\frac{x}{y} \in R_{\frac{a}{b}}$. By arbitrariness of x and y,

$\frac{R_a}{R_b} \leq R_{\frac{a}{b}}$. Conversely suppose z is a value of $R_{\frac{a}{b}}$. Locality of z yields $\frac{a}{b} < z < 0$. Then, $z = k\frac{a}{b}$ for some $0 < k < 1$. Since $\frac{1}{k} > 1$:

$$z = \frac{a}{\frac{1}{k}b} = \frac{\frac{1}{k}a}{\frac{1}{k^2}b}\text{, but } a < \frac{1}{k}a\text{, so that } \frac{1}{k}a \in R_a \quad (31)$$

Similarly, $\frac{1}{k^2}b < b$, so that $\frac{1}{k^2}b \in L_b$. Since z is locally arbitrary, $R_{\frac{a}{b}} \leq \frac{R_a}{R_b}$, establishing:

$$\frac{R_a}{R_b} \leq R_{\frac{a}{b}} \leq \frac{R_a}{L_b} \quad (32)$$

We can proceed a little further by generalizing the situation to monotone continuous functions.

Theorem 14: If f is a function and a in a neighborhood in the domain of f, then:

a) If f is monotone increasing then $f(L_a) = L_{f(b)}$ and $f(R_a) = R_{f(a)}$. Also $f(L_\infty) = L_{\lim_{x\to\infty} f(x)}$ and $f(R_{-\infty}) = L_{\lim_{x\to-\infty} f(x)}$

b) If f is monotone decreasing then $f(L_a) = R_{f(b)}$ and $f(R_a) = L_{f(a)}$. Also $f(L_\infty) = R_{\lim_{x\to\infty} f(x)}$ with $\lim_{x\to\infty} |f(x)| < \infty$ and $f(R_{-\infty}) = L_{\lim_{x\to-\infty} f(x)}$ where $\lim_{x\to-\infty} |f(x)| < \infty$.

Proof: The routine nature of the proof suffices us to mention only part a.

a) Let f be monotone increasing and a is in a neighborhood of its domain. Let $x \in L_a$.

Then $x < a$. Since f is monotone increasing $f(a) < f(x)$. So, $f(x) \in L_{f(a)}$, thus $f(L_a) \leq L_{f(a)}$. Conversely if $y \in L_{f(a)}$, then $y < f(a)$. If $y = f(x)$ then $f(x) < f(a)$.

Since f is monotone increasing, $x < a$, namely $x \in L_a$. So, $y = f(x) \in f(L_a)$, establishing $f(L_a) = L_{f(a)}$. Similar argument applies to show that $f(R_a) = R_{f(a)}$. Now, if f is defined as a neighborhood of ∞. If $x \in L_\infty$ then $x < \infty$.

Monotone increasing property of f yields

$f(x) < \lim_{x\to\infty} f(x)$. Then, $f(x) \in L_{\lim_{x\to\infty} f(x)}$.

Thus, $f(L_\infty) = L_{\lim_{x\to\infty} f(x)}$. Conversely if

$y \in L_{\lim_{x\to\infty} f(x)}$, then $y < \lim_{x\to\infty} f(x)$. If $y = f(x)$, then $f(x) < \lim_{x\to\infty} f(x)$. Since f is monotone increasing, $x < \infty$, namely $x \in L_\infty$. Thus $f(L_\infty) = L_{\lim_{x\to\infty} f(x)}$. Similar argument applies to show that $f(R_{-\infty}) = L_{\lim_{x\to-\infty} f(x)}$.

b) This part is proven in a manner similar to part a. In view of Theorem 13, it is not surprising to have the following theorem.

Theorem 15: Suppose that expression E contains some non-eliminable parameters. If the failure points are not too dense, namely for some $d > 0$, $|a - b| \geq d$, for any failure point a and b, then E may be simplified to contain only one parameter.

Proof: Suppose all parameters in E are to be expressed in R_0. By Theorem 13.d, $L_a = a + L_0$. Invoking Theorem 13.a, $L_0 = -R_0$. Hence $L_a = a - R_0$. Similarly, $R_b = b - L_0$. Now, $L_\infty = \frac{1}{R_0}$ by Theorem 13. Similarly, $R_{-\infty} = \frac{1}{L_0} = -\frac{1}{R_0}$. The parameter R_0 contains non-degenerate interval since $(0, u) \in R_0$ provided $u \leq d$.

In some cases, non-eliminable parameters may be used as an alternative way to find the partial inverse transform of the solutions, specifically for an interval of the form (a, b) where a and b are consecutive failure points. This is because $A = \int_{R_a}^{L_b} k(x)f(x)dx$ implies that:

$$f(x) = \frac{1}{k(x)}\left(\frac{\partial A}{\partial R_a}\right)_{x=R_a} = -\frac{1}{k(x)}\left(\frac{\partial A}{\partial L_b}\right)_{x=L_a} \quad (33)$$

for $x \in (a, b)$.

Finally, an interesting alternative approach to the treatment of this article without invoking new “numbers”, namely in the conference paper [18], and an important progress on the use of Fourier series to vector functions in [19], also a conference paper, witnessing the still actual nature of its application.

4. Conclusions

Instead of the rough categorizing that an integral does not exist, parametric integration gives Fourier series transform a value in the form of an arithmetic expression comprising numbers and parameters. The number of parameters might be reduced to a minimum after some limiting processes.

Although gives a nice array of transform formulas comparable to the Laplace or Fourier transforms, the Fourier series transform is of more restricted use. With parametric integration, it may now handle even more functions than the ordinary Fourier transform, since it can also tackle some functions unsuitable for Fourier integral expansion by freeing it from size requirement $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, as well as with integration troubles at countably many singularity points or divergences at infinity.

Parametric integration is applicable to transforms such as Fourier or Laplace transforms to overcome their shortcomings to the Parametric Integration equipped Fourier Series Transform in order to claim a similar definability and strength.

Solutions of problems that retain non-eliminable parameters can be interpreted as containing some non-standard “numbers,” namely, the parameters that obey arithmetic rules of their own.

When the corresponding failure points are scarce, the non-eliminable parameters can be reduced to only one parameter.

Non-eliminable parameters can be used as an alternative way to find the partial inverse transform of the corresponding expression, namely for the domain part in the form of an interval between two consecutive failure points. For this reason, there might be some cases in which it is desirable to add parameters just for the sake of ease of obtaining inverses even though there is no failure point.

To summarize, the novelty of this article is twofold. First, the introduction of parameters that can add informative details to calculus. Second, the parametric integration technique in handling transforms such as the Fourier series transform, where the parameters are employed either as temporary objects or as parts of the results in the calculations.

Among the prospects, parametric integration might serve as an alternative approach to the study of unbounded self-adjoint linear transformations in functional analysis, which has many applications in partial differential equations. On the other hand, parameters can still be explored and expanded to form more expressive entities.

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