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A Journey into the van der Pol Equation and Haar Wavelet through Engaging Didactic Situations

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Abstract: This qualitative study explores the impact of a didactic sequence based on the theory of didactic situations on the learning of the van der Pol equation and its solution using the Haar wavelet method among undergraduate students in Mathematics and Physics at the Technological University of Pereira. The study objectives focus on identifying students' strengths and challenges, assessing collaborative and individual competencies, and evaluating proficiency in applying the Haar wavelet method to solve the equation. A sample of five students actively participated in all phases of the Didactic Situation, which included action, formulation, validation, and institutionalization. The results indicate that the carefully structured didactic sequence significantly improved students' understanding and engagement with the van der Pol equation and Haar wavelet method. The inclusion of real-world applications and interactive approaches facilitated effective learning strategies. Overall, this study highlights the effectiveness of the theory of didactic situations-based didactic sequence in fostering a deeper appreciation and comprehension of complex mathematical concepts among Mathematics and Physics undergraduates. This research contributes to the enrichment of the academic journey, creating a captivating realm of mathematical beauty and scientific wonder for both students and educators.

Keywords: didactic sequence, the van der Pol equation, the Haar wavelet method, problem-solving.

通过参与教学情境了解范德波尔方程和哈尔小波的旅程

摘要：这项定性研究探讨了基于教学情景理论的教学序列对佩雷拉理工大学数学和物理本科生学习范德波尔方程及其使用哈尔小波方法求解的影响。研究目标侧重于确定学生的优势和挑战，评估协作和个人能力，以及评估应用哈尔小波方法求解方程的熟练程度。五名学生积极参与了教学情境的所有阶段，包括行动、制定、验证和制度化。结果表明，精心构建的教学序列显着提高了学生对范德波尔方程和哈尔小波方法的理解和参与。现实世界的应用程序和交互式方法的纳入促进了有效的学习策略。总体而言，本研究强调了基于教学情境的教学序列理论在促进数学和物理本科生对复杂数学概念更深入的理解和理解方面的有效性。这项研究有助于丰富学术之旅，为学生和教育工作者创造一个数学之美和科学奇迹的迷人境界。

关键词：教学序列、范德波尔方程、哈尔小波方法、问题解决。

1. Introduction

Students enrolled in the Mathematics and Physics Education program focus on a teaching-oriented profile, which necessitates a deep understanding of methodologies and techniques for their future profession. However, it is worth noting that many topics are only superficially covered, such as nonlinear ordinary differential equations and dynamic systems.

The van der Pol equation, which represents a damped oscillator system, is presented only as an example of different types of differential equations. Moreover, students are exposed to a few numerical methods. This equation's interdisciplinary nature, applicable in physics for modeling RLC circuits and in areas such as biology, medicine, and financial administration, makes it an appealing mathematical model for students in the Mathematics and Physics Education program.

In this study, a mathematical education component is introduced, focusing on obtaining the numerical solution of the van der Pol equation (1) using the Haar wavelet method.

$$\frac{d^2X}{dt^2} - \mu(1 - X^2)\frac{dX}{dt} + X = 0 \tag{1}$$

The development of a didactic sequence based on the Theory of Didactic Situations, pioneered by Guy Brousseau, fosters a participative and proactive learning environment.

This study further contextualizes and conceptualizes the implementation of the theory of didactic situations through a-didactic and didactic situations. These phases are framed within a didactic contract defining the rules of engagement in the didactic sequence.

The research chapters outline the problem and research questions, review the state of the art, and establish fundamental concepts of the Theory of Didactic Situations, the van der Pol equation, and the Haar wavelet method. The third chapter focuses on the operational process of equation (1) and its numerical solution using the Haar wavelet method. The fourth chapter delimits the methodological framework for implementing the didactic sequence and the assessment instruments used to evaluate students' learning impact. The fifth chapter presents an analysis of the results obtained after the didactic sequence's implementation, followed by the conclusion of the study.

In summary, this study seeks to enhance students' learning experiences in mathematics and physics education through the implementation of an engaging didactic sequence and the utilization of the Haar wavelet method to tackle the fascinating challenges posed by the van der Pol equation.

2. Justification

The teaching and learning processes in various knowledge disciplines primarily focus on constructing and generating logical-deductive thinking to solve problems. In the field of sciences, mathematical reasoning plays a crucial role in creating models representing the studied phenomena, leading to meaningful learning experiences for students. However, many topics, such as nonlinear ordinary differential equations and dynamic systems, are only superficially taught and lack operational depth.

The van der Pol equation, a nonlinear ordinary differential equation representing a damped oscillator system, is merely presented as an example without delving into its full applicability in areas such as biology, physics, medicine, and financial administration. To bridge this gap, this study implements a didactic sequence based on the theory of didactic situations developed by Guy Brousseau. This will enable students in the Mathematics and Physics Education program at the Technological University of Pereira to gain a deeper understanding of the van der Pol equation and its numerical solution using the Haar wavelet method.

From the perspective of mathematics education, it is evident that students often solve mathematical problems mechanically to obtain correct answers. However, abstraction in the mathematical modeling process poses a challenge for students to conceptually grasp and solve real-world problems. Traditional teaching methods with vertical interaction between teachers and students contribute to this issue. The didactic sequence from the theory of didactic situations avoids such traditional processes and fosters a more interactive and participative learning environment.

This study intends to strengthen the interaction between the teacher, student, and mathematical knowledge related to the van der Pol equation (1). By promoting a proactive and participative role, students are encouraged to authentically comprehend and engage with mathematical knowledge. The goals are to empower students to face problems independently and develop autonomous and meaningful learning habits aligned with their real-life context.

The implementation of the didactic sequence, organized into four interconnected moments in the classroom, enables students to relate concepts of the van der Pol equation and determine its behavioral structure. This fosters cognitive self-reflection regarding the mathematical model. By analyzing the learning impact of this didactic approach, this study seeks to show the significance of integrating the theory of didactic situations to create a meaningful and relevant learning experience for students in the Mathematics and Physics Education program.

3. State of the Art

The state of the art is crucial for the current thesis, as it highlights previous research related to the theory of didactic situations, the van der Pol equation, and the Haar wavelet method. This literature review aims to identify existing works that can contribute to the development of a didactic sequence for learning Equation (1).

3.1. Theory of Didactic Situations

Although the cited undergraduate theses do not directly relate to the purpose of implementing a didactic sequence for learning the van der Pol equation, they are essential for supporting the methodology of mathematics education in the four phases of a didactic situation from Brousseau's perspective. The works listed below are relevant to this study:

Insuasty (2014) conducted qualitative descriptive research titled "Changes Produced in the Learning of Quadratic Function Families in Ninth Grade Students through Didactic Situations Using Geogebra." This research is based on Brousseau's Theory of Didactic Situations and shows how implementing didactic situations can promote dynamic and autonomous performance in students, enabling them to grasp and apply quadratic function concepts.

At the international level, [2] and [3] have contributed to the development and understanding of the theory of didactic situations. Their works focus on the application of didactic situations in different contexts and emphasize the importance of avoiding traditional teaching methods and fostering independent learning.

In Peru, some studies have explored the implementation of the theory of didactic situations in teaching probabilities and other mathematical concepts to secondary school students. [4] illustrate the potential improvements in learning using this theory.

Regarding the van der Pol Equation, most studies focus on its application in the context of oscillators and circuit theory. Here are some relevant references:

[5] explored the teaching and analysis of simple harmonic motion, which serves as a foundation for understanding the Van Der Pol Equation.

[6] wrote about the historical and problem-oriented approach to teaching ordinary differential equations (ODEs), with a specific focus on the van der Pol oscillator.

Luis Fernando Huertas (2014) proposed a didactic approach for teaching damped motion based on active learning and video analysis. Although the pedagogical approach is not directly related to Brousseau's theory, the mathematical and conceptual aspects of damped oscillation are covered.

[7] presented a proposal for approaching oscillations and waves at the university level, which includes the van der Pol Equation as one of the topics.

Works conducted at the local level at the University of Pereira have mainly focused on the design and implementation of applications and prototypes for teaching simple harmonic motion and damped oscillation, often combining technology with mathematical concepts.

The Haar wavelet method is commonly used in signal processing applications. However, there are some studies directly related to its applicability in the numerical solution of differential equations. The method has been used in both ordinary and partial differential equations. Here are some relevant references:

[8] presented a detailed description of the Haar wavelet method's application in solving lumped and distributed-parameter systems of ordinary differential equations. They proposed an algorithm for calculating the operational matrix of Haar, which is valuable for numerical solutions. [9] provided a numerical solution for the L-C-R series equation using the Haar wavelet method. This paper discusses the mathematical model representing the RLC circuit and its numerical solution using the Haar wavelet method. [10] described the application of the Haar wavelet collocation method for numerically solving singular initial value problems.

Lepik [11-14] has made significant contributions to the use of the Haar wavelet method for approximating and solving various mathematical models.

At the local level in Pereira, the University Tecnológica de Pereira, there are not many works focused on using the Haar wavelet method for the numerical solution of differential equations. The only relevant work found is by Magister María Clemencia Guerrero Ruales (2022), as mentioned previously, where the Haar wavelet method is used for the numerical solution of the one-dimensional nonhomogeneous heat equation.

In conclusion, while the Haar wavelet method is more commonly associated with signal processing applications, some studies at the international and national levels have explored its applicability in the numerical solution of differential equations. At the local level, limited research is focused on this specific application.

4. Theoretical Framework

This research is based on the implementation of a didactic sequence for learning the van der Pol equation and its numerical solution using the Haar wavelet method. The theoretical foundation for structuring the didactic sequence is the theory of didactic situations, developed by Guy Brousseau, a French mathematician and researcher. The following are the key conceptual elements of this theory:

4.1. Psychopedagogical Foundation

The theory of didactic situations examines how mathematical thinking is appropriated. Brousseau's theory is grounded in constructivism, which suggests that true human learning involves constructing knowledge within oneself. The learner undergoes cognitive disequilibrium when facing a problem situation and then constructs new knowledge to resolve it. In this process, the learner plays an active role, and the teacher serves as a guide and facilitator of the learning conditions.

4.2. The Influence of Constructivist Psychology

Brousseau's theory is influenced by constructivist psychology, particularly Piaget's genetic epistemology. According to Piaget, humans progress through various stages of development, from the sensorimotor phase to formal operational thinking. Brousseau incorporates this view into his theory, recognizing that learners bring their preconceptions to the learning situation and undergo cognitive transformations to generate new knowledge.

Brousseau's theory is influenced by Jean Piaget's genetic psychology, which asserts that learners adapt to their environment by assimilating and accommodating new information. Learning involves constructing knowledge through the assimilation and accommodation of new events or information.

4.3. Didactics

Didactics is the backbone of Brousseau's theory of didactic situations. It refers to conceptual and procedural actions related to teaching, planning, curriculum development, and learning. Brousseau views didactics as a discipline that describes and studies teaching activities within a specific scientific discipline. It also involves creating instructional alternatives for the teaching and learning processes.

4.4. Didactics of Mathematics

Didactics of mathematics is a specific field within didactics that focuses on the teaching and learning processes of mathematics. It provides conceptual, methodological, and procedural tools to enhance learning. Brousseau distinguishes between two approaches in mathematics didactics: one focuses on fundamental didactics (the French school) and the other on significant education. Brousseau adopts a significant learning approach that emphasizes meaningful learning experiences for students.

4.5. Learning

Brousseau's theory incorporates David Ausubel's significant learning concept, which emphasizes meaningful learning experiences. Learning occurs when learners assimilate new information into their existing knowledge, generating meaningful connections and understanding.

4.6. Situations

Situations play a significant role in Brousseau's theory. He considers a situation to be an interaction scheme between a learner and their environment. In this interactive process, learners construct new knowledge, using the learning situation to create new understanding and meaning.

4.7. Silent Actor

Brousseau refers to a silent actor, which is the interaction between the teacher, the learner, and the knowledge within a situation. This interaction occurs in a context facilitated by a didactic contract that outlines the rules and expectations within the learning environment.

5. Theory of Didactic Situations

This research work focuses on the implementation of a didactic sequence based on the theory of didactic situations by Guy Brousseau, a mathematician and researcher born on February 4, 1933, in Taza, Morocco. Brousseau developed a great passion for mathematics, which led him to create his theory.

The theory of didactic situations is based on Piaget's genetic psychology and the theoretical foundations of constructivism and meaningful learning. It emphasizes the interaction between the teacher, the student, and mathematical knowledge, mediated by the didactic contract.

This theory divides the didactic situation into four phases: action, formulation, validation, and institutionalization. The teacher presents problems to the students, who interact with the didactic environment and construct their knowledge as they solve the given situation.

The didactic situation is distinguished from the a-didactic situation, in which the student acquires knowledge independently, without direct intervention from the teacher. Brousseau encourages teachers to motivate students to explore, make conjectures, and verify their solutions, allowing them to construct knowledge by themselves.

In summary, Brousseau's theory of didactic situations emphasizes the dynamic interaction between the teacher, the student, and mathematical knowledge, promoting independent and meaningful learning. By creating conducive learning environment, teachers can facilitate students' active and meaningful engagement with mathematics.

It is important to understand the concepts of a didactic contract and didactic situation, which form the basis for the implementation of the didactic sequence. The didactic contract establishes the rules of the didactic relationship, while the didactic situation provides a context for the interaction between the teacher, the student, and mathematical knowledge.

The four phases of the didactic situation – action, formulation, validation, and institutionalization – guide the progression of the didactic sequence. In the action phase, the teacher presents the problem, and the student begins to explore and formulate initial ideas. The formulation phase involves active participation and communication among students as they work collaboratively or individually on the problem. In the validation phase, the teacher acts as a mediator, providing feedback and assessing the students' work. Finally, in the institutionalization phase, the teacher formalizes the acquired knowledge, and the students compare their solutions to established mathematical concepts.

The a-didactic situation, on the other hand, refers to moments when students work independently and rely on their prior knowledge to approach the problem-solving process. The teacher acts as a facilitator, guiding students without providing explicit directions or expectations.

By understanding these concepts and phases, educators can design effective didactic sequences that foster meaningful mathematical learning experiences. The emphasis on student engagement, exploration, and collaboration aligns with constructivist principles, promoting active learning and a deeper understanding of mathematics.

6. Mathematical Foundations of the van der Pol Equation

The study of the van der Pol equation (1) as a mathematical object for learning by students in the Mathematics and Physics program at the University of Technology of Pereira, using a didactic sequence based on the theory of didactic situations, involves dealing with a non-linear ordinary differential equation that represents a dynamic system. The following are the theoretical and procedural concepts necessary to approach this type of differential equation:

The van der Pol equation is solved using a numerical method called the Haar wavelet, which uses a family of functions with specific properties. To define this family in a vector space with the appropriate structure, we use a Hilbert space.

The Hilbert space, denoted by $\mathcal{H}$, is a vector space equipped with a metric and an inner product such that, for two functions f and g belonging to the space, their inner product is defined as follows [15]:

$$< f, g >= \int f(x)\overline{g(x)}dx \ , f, g \in \mathcal{H} \qquad (2)$$

Moreover, as mentioned by [15], the Hilbert space with its metric is a "complete vector space." In particular, if $f \in L^2(\mathcal{R}) \subset \mathcal{K}$, the inner product of f with itself is a positive number, given that f is square-integrable. This means that the integral of the square of the absolute value of f is finite, satisfying the following property:

$\int |f(x)|^2 dx < \infty$.

To conceptualize the nonlinear harmonic oscillator, which essentially corresponds to the system represented by the van der Pol equation, it is essential to begin with the description of a simple system: a point mass attached to the end of an ideal spring with no mass and a spring constant K. In this setup, the mass oscillates around the equilibrium point of the spring, and for simplicity, we assume that this motion is in a single direction, in this case, along the horizontal direction x, as depicted in Fig. 1.

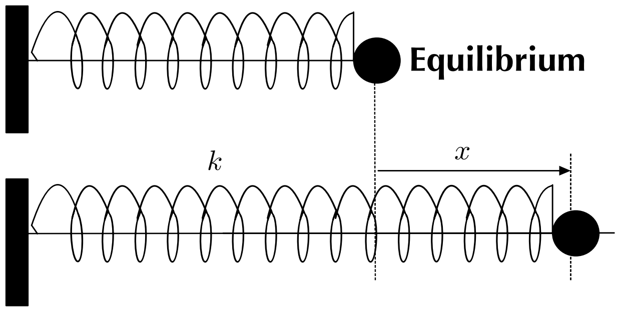

Fig. 1 Mass-spring system (Developed by the authors)

The differential equation that governs is

$$\frac{d^2x}{dx^2} + kx + \lambda\frac{dx}{dt} = 0, \qquad (3)$$

and its solution is given by

$$x(t) = e^{-\lambda t}[A\cos(\sqrt{\frac{k}{m} - \lambda^2 t} + B\,\mathrm{sen}\sqrt{\frac{k}{m} - \lambda^2 t}.$$

7. The Haar Wavelet

7.1. Numerical Method

The most common use of the Haar wavelet is evident in signal processing; however, through its definition and operability, it has become a numerical method for providing approximate solutions to both ordinary and partial differential equations. Historically, the Haar wavelet draws on the concepts of Fourier and its operational bases in signal treatment to structure its mathematical framework for various applications. It is important to mention that there is a wide variety of wavelets, each family with its respective algorithms and procedural considerations, with many proponents for each class of wavelet. However, their genesis can be appreciated in the following observation:

The concept of wavelets in its current theoretical form was first proposed by Jean Morlet and colleagues at the Marseille Theoretical Physics Center (France). Wavelet analysis methods were mainly developed by Y. Meyer and collaborators, and the main algorithms date back to the work by Stephane Mallat of 1988 [16].

Now, focusing on the Haar wavelet, which is part of the mathematical object of this research, we reference the Hungarian mathematician of Jewish origin Alfred Haar as its creator. In his thesis of 1909, he formally introduced wavelet theory, and in his honor, the Haar measure, Haar wavelet, and Haar transformation are recognized. Alfred Haar was born in Budapest on October 11, 1885, and died in Szeged on March 16, 1933. He is also recognized for, along with Frigyes Riesz, turning the University of Szeged into a center for mathematics and founding the journal "Acta Scientarum Mathematicarum."

As a historical context for the Haar wavelet, it is necessary to mention Ingrid Daubechies (one of the classes of wavelet is named after her) because she created an algorithm for constructing functions with compact support, providing the so-called "h" filters, which are orthogonal and give the property of "null moments." These types of functions are extremely relevant for the usability of wavelets, as mentioned by [17]: "These functions are characterized by being orthogonal, having compact support, different degrees of smoothness, and the maximum number of null moments in the time domain, depending on their width," which ensures a high level of reliability in their use in wavelet operations. The Daubechies wavelets are symbolized as "dbN," where "N" represents the order, with "N" being a positive integer; when N = 1, they are known as the Haar wavelet (Daubechies of Order One: db1). The Haar wavelet is particularly didactic due to its simplicity, compactness, and symmetry, although it has the limitation of discontinuity.

7.2. The Haar Function

The Haar wavelet method relies on specific functions that provide the necessary mathematical foundation. For this reason, the so-called Haar function is defined. Let $\Phi \in L^2(R)$:

$$\Phi(t) = \begin{cases} 1, 0 \leq t < \frac{1}{2} \\ 0, elsewhere \end{cases}.$$

The function Φ is known as the scaling function.

Furthermore, a set of scaling functions is defined as follows:

$$\Phi_k(t) = \Phi(t-k) = \begin{cases} 1, t_1 = k \leq t < k+1 = t_2 \\ 0, elsewhere \end{cases},$$

where k is an integer. This allows any function f to be expressed as a linear combination of the scaling functions $\Phi_k(t)$ with their respective coefficients c_k. To calculate these coefficients, the scaling functions must satisfy the orthonormality property.

7.3. The Haar Wavelet Function

The Hilbert space is considered suitable for the family of scaling functions in the subspace $V0$, especially when considering their application in signal processing. However, various studies have shown that it is not favorable to expand the space of scaling functions to improve signal analysis. Instead, it is more appropriate to define a new space V_1 as a complement. In this new space, the following functions are defined:

$$\Psi(t) = \begin{cases} 1, 0 \leq t < \frac{1}{2} \\ -1, \frac{1}{2} \leq t < 1 \\ 0, elsewhere \end{cases}.$$

This function is known as the Haar wavelet function and $\Psi(t) \in L$.

The graph illustrates that the Haar wavelet function is defined on the closed interval [0,1].

Similarly, the family of functions can be expanded as follows:

$$f(t) = \sum_{k \in Z}^{\infty} d_k \Psi_k(t), -\infty < k < +\infty$$

If the orthonormality property of $\Psi(t)$ is satisfied, the coefficients dk can be calculated as follows:

$$d_k = < f(t), \Psi_k(t) > = \int_{t_1}^{t_2} f(t)\Psi k(t)dt$$

This Haar wavelet function can be interpreted as a sampling in the use of the Haar wavelet transform, as translations and dilations can be applied one by one according to the level of resolution used. Thus, the Haar function, also known as the Haar wavelet (in honor of its creator Alfred Haar, 1909), uses translations and dilations.

7.4. The Haar Matrices

When using the Haar wavelet method to solve ordinary or partial differential equations, it is possible to do so procedurally in two ways. One is to solve the system of linear equations that result in finding the Haar wavelet coefficients "ai." The other approach is to use matrix operations, which saves computation time when higher resolutions are used, as it leads to systems of linear equations of the order $8X8$, $16X16$, $32X32$, etc. Consequently, it is possible to represent the Haar function in its respective matrix, considering the implemented resolution level.

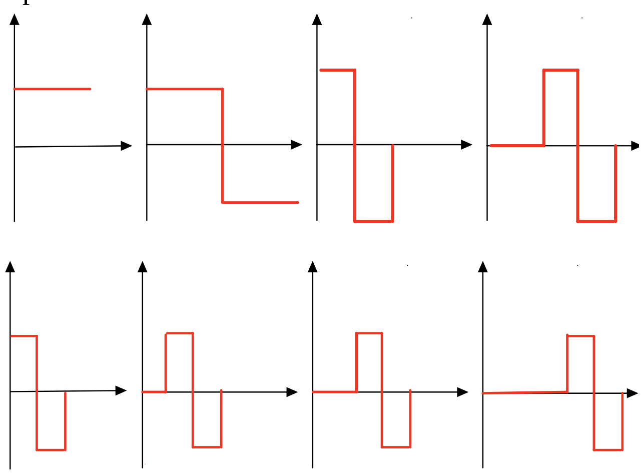

Fig. 2 Family of the Haar functions [1]

[11] generalized the family of the Haar functions as

follows:

$$h_i(t) = \begin{cases} 1, \frac{k}{m} \leq t < \frac{k+\frac{1}{2}}{m} \\ -1, \frac{k+\frac{1}{2}}{m} \leq t < \frac{k+1}{m} \\ 0, elsewhere \end{cases}$$

The first Haar basis obtained when $i = 1$ is called the mother wavelet, and when $i = 2$, it is called the daughter wavelet. Subsequent bases are obtained through translation and dilation operations, resulting in other Haar bases.

The following are some examples of the Haar matrices.

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Furthermore, the Haar wavelet functions have compact support in the closed interval $[0,1] \subset L$.

They are also orthonormal functions, unlike other wavelets that do not possess this property. However, the Haar wavelets are piecewise constant functions, which makes them unsuitable for direct application in solving differential equations because of their discontinuity, which hinders differentiability in the interval.

To overcome this difficulty, researchers proposed expanding the higher-order derivative of the given ordinary differential equation into the Haar series. This approach can be found in [8]. In this study, they also established the discretization of the system using collocation methods.

7.5. General Description of the Wavelet Haar Numerical Method

The Haar wavelet numerical method is a simple approach to solving ordinary or partial differential equations. In this case, we deal with a nonlinear ordinary differential equation. However, the general steps for using the method to solve differential equations remain the same, making it easier to understand. According to [18], let us consider the general form of an ordinary differential equation:

$$A_1 \frac{d^n x}{dt^n} + \frac{A_2 d^{n-1} x}{dt^{n-1}} + \frac{A_3 d^{n-2} x}{dt^{n-2}} + \cdots + A_n \frac{dx}{dt} + A_{n+1} x(t) = f(t) \qquad (4)$$

The steps to solve it using the Haar wavelet method are as follows:

i) Define the highest-order derivative in the equation as a Haar series expansion:

$$x^{(n)}(t) = \sum_{i=0}^{\infty} a_i h_i(t), \qquad i \in \{0\} \cup \mathcal{N}$$

ii) Integrate the Haar series expansion from Equation (17) repeatedly to obtain the Haar series expansions for the lower-order derivatives in the differential equation until an expression for the function $x(t)$ is obtained:

$$x^{(n-1)}(t) = \int_0^t \left(\sum_{i=0}^{\infty} a_i h_i(t) \right) dt$$

$$x(t) = \int_0^t \ldots \int_0^t \left(\sum_{i=0}^{\infty} a_i h_i(t) \right) dt \qquad (5)$$

iii) Replace the Haar series expansion for the highest-order derivative and the expressions for the other derivatives obtained in Step Two into the differential equation. Then, evaluate them at the collocation points according to the level of resolution defined for the Haar wavelet method, using the following expression:

$$t_l = \frac{l - \frac{1}{2}}{2m} \qquad (6)$$

where $m = 2j$, with j = 0, 1, 2, …, where J indicates the maximum level of resolution of the Haar wavelet. Also, l = 1, 2, 3, … , $2m$.

iv) Calculate the Haar wavelet coefficients ai by solving the system of equations generated in the expression resulting from Step Three. This can be done using either an algebraic or a matrix-based approach.

v) Replace the obtained Haar wavelet coefficients ai into the expansion generated for the function (t) according to expression (18). Finally, determine the numerical approximations as solutions to the differential equation for the selected level of resolution.

For example, for $\mu = 1$, in (1),

$$\frac{d^2X}{dt^2} - (1 - X^2)\frac{dX}{dt} + X = 0 \qquad (7)$$

From (6), taking into account $t_1 = \frac{1}{8}$, $t_2 = \frac{3}{8}$, $t_3 = \frac{5}{8}$, and $t_4 = \frac{7}{8}$,

$$\frac{d^2x(t)}{dt^2} = \sum_{i=0}^{4} a_i h_i(t), \frac{dx}{dt} = \int_0^t \left(\frac{d^2x(t)}{dt^2}\right) dt$$

$$= \int_0^t a_i h\_i(t)\, dt = \sum_{i=0}^{\infty} a_i p_i(t)$$

$$x(t) = \sum_{i=0}^{\infty} a_i q_i(t) \qquad (8)$$

Taking into account these expressions, we have

$$\sum_{i=0}^{\infty} \mathrm{a_i h_i}(t) - \left[1 - \left(\sum_{i=0}^{\infty} \mathrm{a_i q_i}(t)\right)^2\right]\left(\sum_{i=0}^{\infty} \mathrm{a_i p_i}(t)\right) + \sum_{i=0}^{\infty} \mathrm{a_i q_i}(t) = 0$$

$$\sum_{i=1}^{4} \mathrm{a_i} \left[\mathrm{h_i(t) - p_i(t) + a_i^2 \left(q_i(t)\right)^2 p_i(t) + q_i(t)}\right] = 0$$

From this, we find the Haar wavelet coefficients a_i from the following system:

$$\mathrm{a_1^2 + a_2^2 + a_3^2 = 84.992;}$$

$$\mathrm{a_1^2 + a_2^2 + (0.020164609)a_3^2 = 17.27736692,}$$

$$\mathrm{(0.023841857)a_1^2 + (0.012107849)a_2^2 = 0.3203125;}$$

$$\mathrm{(0.12822733)a_1^2 + (0.007331848145)a_2^2 + (0.000373840332)a_4^2 = 0.305664062,}$$

i.e., $\mathrm{a_1 = 3.302712652,\ a_2 = 2.30676306,\ a_3 = 8.313132524,\ a\_4 = 2.157923759.}$

Table 1 Numerical solutions for x(t) using the Haar wavelet method with μ = 1 (Developed by the authors)

$\frac{\mathbf{t_l}}{\mathbf{8}}$	$\boldsymbol{x(t)}$ **(current in the circuit modeled by (1))**
1	2.146487875268
3	2.632149744321
5	2.001863358944
7	1.100854692673

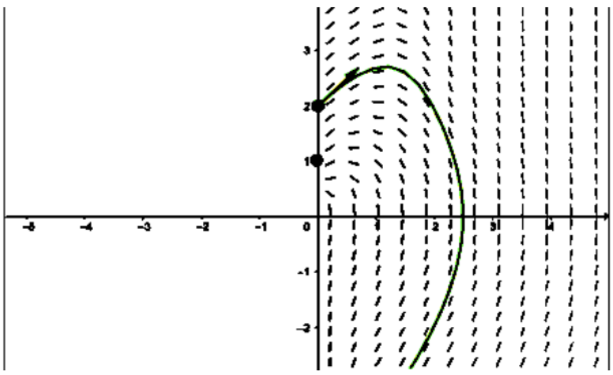

Fig. 3 Phase plane of the van der Pol equation for a resolution level equal to one (Developed by the authors)

In the same way, for a resolution level of three, we have:

Table 2 Numerical solutions for x(t) using the Haar wavelet method with μ = 1 (Developed by the authors)

$a_1 = 0.56254871$	$a_2 = 1.25484621$	$a_3 = 1.8265317$	$a_4 = 1.9876458$
$a_5 = 2.0685832$	$a_6 = 2.1986324$	$a_7 = 2.6115271$	$a_8 = 3.6939732$
$a\_9 = 5.6464812$	$a_{10} = 8.1914351$	$a_{11} = 7.4634276$	$a_{12} = 6.0826931$
$a_{13} = 4.3737628$	$a_{14} = 1.8359467$	$a_{15} = 0.7683574$	$a\_16 = 0.3734612$

$\frac{\boldsymbol{t_l}}{\mathbf{32}}$	$\boldsymbol{x(t)}$
1	1.033.721.564.274
3	2.132.149.744.321
5	2.274.721.400.664
7	2.366.147.522.871
9	2.611.590.143.322
11	28.558.297.737.158
13	29.000.120.310.431
15	29.015.414.711.133
17	28.871.512.433.016
19	20.001.501.432.374
21	17.666.000.489.221
23	13.534.315.068.574
25	100.495.235.178.056
27	0.99948736461023
29	0.41358090735667
31	0.00048765959871

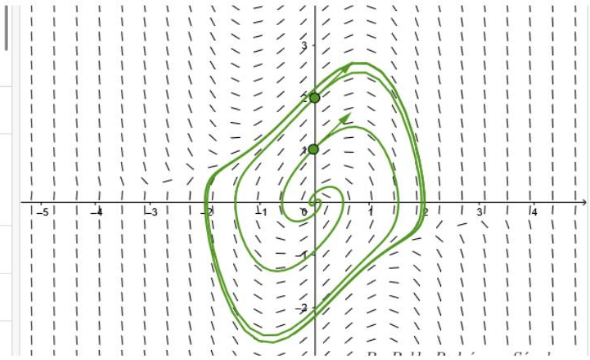

Fig. 4 Phase plane of the van der Pol equation for a resolution level equal to three (Developed by the authors)

8. Didactic Sequence for Learning the van der Pol Equation and Its Solution Using the Haar Wavelet Method

8.1. Initial Interaction with the Problem Situation (Action Situation)

- The teacher opens the session and explains the activities to the students, setting the guidelines for the three sessions.

- The teacher presents a problem situation related to an object attached to a spring and its behavior. The students share their ideas and positions on the topic.

- The teacher introduces the problem situation of the van der Pol ordinary differential equation, and the students express their initial conceptions about it.

8.2. Experimental Approach to the Mathematical Model of the van der Pol Equation (Formulation Situation)

- The students work in groups to experiment with a mass-spring system and gain insights into harmonic oscillators as a prelude to understanding the nonlinear oscillator represented by the van der Pol equation.

- The teacher provides guidelines and materials for the laboratory experiment, where students record data for analysis.

- The students share their prior knowledge and conceptual positions regarding the van der Pol equation through group work.

8.3. Procedural Confrontation of the van der Pol Equation and the Haar Wavelet Method (Validation Situation)

- In the second session, the teacher initiates a reflection on the laboratory experiment.

- The teacher introduces the conceptualization of linear harmonic oscillators and the RLC circuit as the physical and mathematical foundation of the van der Pol equation.
- The derivation of the van der Pol equation and an overview of the Haar wavelet method are provided by the teacher.
- The students receive a guide to self-reflect on the intervention and confront the mathematical model and the Haar wavelet method.

8.4. Knowledge Formalization (Institutionalization Situation)

- The third session facilitates a dialog of knowledge among the teacher, students, and the learned concepts of the van der Pol equation and the Haar wavelet method.
- A conversational approach is used to encourage the students' active participation.
- The teacher provides feedback on the entire process and collaborates with the students to formalize the mathematical knowledge acquired.
- The session concludes with a guide for the students to review their learning, closing the didactic sequence.

Throughout the sequence, the teacher acts as a guide and facilitator, encouraging students to take an active role in consolidating their knowledge and understanding the van der Pol equation and its solution using the Haar wavelet method. The didactic sequence is structured in three sessions of 2 h each, with a focus on engaging the students and promoting their active participation in the learning process.

9. Results

The students who participated in the didactic sequence voluntarily accepted to be part of the three sessions for its implementation. Throughout the process, the students actively and willingly engaged, which allowed for an objective, clear, and pertinent development of the research, aligned with the proposed objectives. Guided by specific rubrics for each phase (action, formulation, validation, and institutionalization), the students' performance was documented and assessed through the use of guides that were consistent with the intention of each phase. The guides highlighted the a-didactic situation as a student learning scenario.

In this chapter, the results obtained are presented using the established categorization in the assessment rubrics. The corresponding analysis was conducted to determine the impact of the implementation of the didactic sequence on the students' learning of the van der Pol equation and its solution using the Haar wavelet numerical method.

In this qualitative research, the results were analyzed using the content analysis approach. This method involves observing and categorizing the verbal and written participation of each student in the sample. The analysis focused on the performance of the students in each phase of the didactic situation, aligned with the implemented didactic sequence. The students' performance was assessed using rubrics for each phase (action, formulation, validation, and institutionalization), and the results were presented in a graphical format to illustrate the group's overall performance in learning the van der Pol equation and its solution using the Haar wavelet numerical method.

9.1. Analysis of the Initial Interaction Phase (Action)

During the initial interaction phase, the students individually completed a questionnaire to assess their prior knowledge. The responses to the introductory reading showed that the students had a general understanding of linear proportional relationships between variables. However, weaknesses were identified in algebraic manipulations, limits, integrals, and operations with matrices, which are essential for understanding the van der Pol equation. Furthermore, the students showed limited conceptual understanding of the RLC circuit, which is the basis for modeling the van der Pol equation.

The results indicate that the students had weaknesses in their prior knowledge, particularly in descriptors A1, A2, A3, and A4, demonstrating limited conceptual and procedural foundations for approaching the van der Pol equation.

9.2. Analysis of the Experimental Approach Phase (Formulation)

In the experimental approach phase, the students worked in groups to perform a laboratory practice involving an oscillator system. Percentages of the students' performance are reflected in each descriptor of the formulation phase. The results indicate that the students performed well in the laboratory practice, with 60% achieving an excellent score. Only descriptor F4 showed weaknesses, with 10% of the students rated insufficient.

9.3. Analysis of the Procedural Confrontation Phase (Validation)

In the procedural confrontation phase, the students engaged in a conversation to consolidate their knowledge of the mathematical model of the van der Pol equation and its numerical solution using the Haar wavelet method. The results indicate that the students showed improved performance with no insufficiency. However, some weaknesses remained in the regular category, indicating the need for further reinforcement in certain aspects of the mathematical model and numerical method.

9.4. Analysis of the Institutionalization Phase

In the institutionalization phase, the students' knowledge was reviewed through a formal questionnaire. Percentages of the students' performance are reflected in each descriptor of the institutionalization phase. The results indicate that 90% of the students achieved an excellent score with no insufficiency. The students' responses demonstrated a strong understanding of the subject matter, confirming the success of the didactic sequence in facilitating their learning of the van der Pol equation and the Haar wavelet numerical method.

The implementation of the didactic sequence proved to be effective as it engaged the students actively, improved their understanding, and led to successful learning outcomes. The results demonstrate that the students' performance improved significantly throughout the different phases of the didactic situation.


④ $f(x) = \frac{3}{4}x^5 + 3x^3 - 5x^2 + sen(x) - 2\cos(x) +$

$f'(x) = \frac{15}{4}x^4 + 9x^2 - 10x + \cos(x) + 2\,sen(x)$

$f''(x) = \frac{60}{4}x^3 + 18x - 10 - \cos(x) + 2\cos(x)$

$f'''(x) = 45x^2 + 18 + sen(x) - 2sen(x)$

⑤ $\int (4x^3 + 3e^{-3x} - sen(x))\,dx$

$\int 4x^3 dx + \int 3e^{-3x} dx - \int sen(x) dx$

$4\int x^3 dx + 3\int e^{-3x} dx - (-\cos(x))$

$4\frac{x^4}{4} + 3\frac{e^{-3x}}{-3} + \cos(x)$

$= x^4 - e^{-3x} + \cos(x)$

9. Explique con sus palabras cual es la definición de oscilación.

es un movimiento que se considera como periodico porque cierto intervalo de tiempo pasa por la posicion de equilibrio

10. Menciona tres ejemplos de situaciones de la cotidianidad en que se presenta un movimiento oscilatorio:

Un columpio
un resorte
la viela de un motor.

11. Explica con tus palabras en que consiste un sistema conocido como oscilador lineal y no lineal:

un oscilador lineal es por ejemplo un sistema masa-resorte que se desplaza en una sola direccion. No lineal es que se sale de la trayectoria

Fig. 5 Answers to the mathematical component of the prior knowledge review guide. These responses were provided by one of the students in the sample group regarding the topics necessary for learning the van der Pol equation from a mathematical perspective (Own elaboration)

10. Conclusions

i) The conceptual and contextual structuring established for learning the ordinary nonlinear differential equation, known as the van der Pol equation, was highly significant in effectively delimiting the "problem situation" presented to undergraduate students majoring in mathematics and physics. This allowed them to identify their strengths and weaknesses in the initial stage (action situation), thereby creating appropriate strategies for the timely implementation of the didactic sequence.

ii) The inclusion of laboratory practice in the formulation phase enabled the students to strengthen their learning by putting their knowledge into action and sharing it through assertive communication with their peers. This approach facilitated their appropriation of the conceptual and procedural elements necessary for learning the van der Pol equation through experimentation.

iii) The incorporation of a conversational approach and the use of a guide in the validation phase allowed each student to grasp different concepts, operational processes, and the application of the Haar wavelet method for the numerical solution of the van der Pol equation. This resulted in a satisfactory performance when applying the method. Moreover, the conversational approach encouraged students to actively contribute and enrich the learning process for themselves and their peers.

iv) The development of the institutionalization phase through a conversational approach, with the teacher-researcher acting as a moderator and facilitator, and proactive and purposeful participation of each Mathematics and Physics major allowed the demonstration of well-founded formal arguments concerning the mathematical model of the van der Pol equation and use of the Haar wavelet method for solving differential equations. Furthermore, the teacher-researcher was able to determine, through the final review of acquired knowledge, that tracing each student's performance during the implementation of the "didactic sequence" had a positive impact on the incidence of this approach in the learning of the mathematical model under study.

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