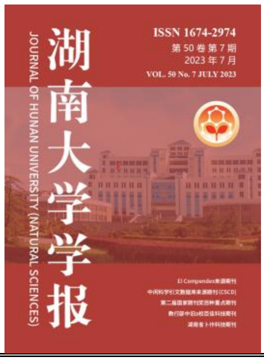

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European Call Option and the Hamilton-Jacobi Type in Classical and Fractional Forms

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Abstract: This study aims to investigate the European Call Option in a fractional way and the reaction of the Hamilton-Jacobi differential equation with the European Call Option. We explore the applications of Mellin transforms as a method for solving such fractional differential equations. We can address initial value problems defined for arbitrary orders of differentiation and integration in fractional calculus that allows us to understand the solution's evolution, encompassing the values established in classical analysis. The theory of options permits the connection of physics and finance, particularly through dynamic systems and their relationship with the Hamilton-Jacobi equation. The European Call Option model exhibits energy in its Hamiltonian, leading to investigations of the Lagrangian and its potential functions within the European Call Option model. Caputo's and Riemann-Liouville's fractional derivatives are crucial for differential equations with integer and non-integer initial conditions, respectively. This new perspective on calculus challenges traditional analysis, explores new horizons, and demands a redefinition of classical approaches. Understanding the efficacy of integro-differential operators is pivotal for advancing research, particularly in interpreting financial phenomena using classical calculus. The Mittag-Leffler function proves valuable for algebraic purposes and inverse transformations such as Mellin.

Keywords: European call option, the Hamilton-Jacobi equation, the Mittag-Leffler function, the Mellin transform, the Riemann-Liouville fractional derivative.

欧式看涨期权以及经典和分数形式的汉密尔顿-雅可比类型

摘要：本研究旨在以分數方式研究歐式看漲期權以及漢密爾頓·雅各比微分方程與歐式看漲期權的反應。我們探索梅林變換作為求解此類分數階微分方程的方法的應用。我們可以解決分數階微積分中為任意階微分和積分定義的初始值問題，這使我們能夠理解解決方案的演變，包括經典分析中建立的值。期權理論允許物理學和金融的聯繫，特別是通過動態系統及其與漢密爾頓雅可比方程的關係。歐式看漲期權模型在其哈密頓量中展現了能量，從而引發了對拉格朗日量及其在歐式看漲期權模型中的潛在功能的研究。卡普托和黎曼劉維爾的分數階導數分別對於具有整數和非整數初始條件的微分方程至關重要。這種微積分的新視角挑戰了傳統分析，探索了新的視野，並要求重新定義經典方法。了解積分微分算子的功效對於推

進研究至關重要，特別是在使用經典微積分解釋金融現象方面。事實證明，米塔格-萊弗勒函數對於代數用途和梅林等逆變換很有價值。

关键词：欧式看涨期权、汉密尔顿-雅可比方程、米塔格-莱弗勒函数、梅林变换、黎曼-刘维尔分数阶导数。

1. Introduction

Options are extensively used in economic contexts. They provide a flexible approach to managing risks and can be treated as valuable financial instruments. Trading strategies based on options have become increasingly popular, challenging, and profitable.

The theory of options plays a vital role in securities transactions and economics because it can model various business processes. The connection between physics and finance is evident in the dynamic system of options, which is related to the Hamilton-Jacobi equation. The European Call Option model possesses an inherent energy represented by its Hamiltonian, which opens avenues for studying the Lagrangian and potential functions within the Black-Scholes model. In differential equations with integer initial conditions, the Laplace transform of Caputo's fractional derivative is essential. However, when dealing with cases that do not have integer initial conditions, it is recommended to use the Laplace transform of the Riemann–Liouville fractional derivative for polynomial and trigonometric functions, inheriting properties from classical calculus. This new perspective on calculus deviates from tradition, exploring new horizons and demanding a redefinition of classical analysis. Understanding the effectiveness of integro-differential operators is crucial for advancing this research. The behavior of functions under arbitrary parameters contributes to a broader interpretation of financial phenomena using classical calculus. For algebraic purposes and inverse transformations such as Laplace and Fourier, the Mittag-Leffler function is highly helpful for solving initially complex problems.

Under the conditions of the market economy, the stock market is becoming increasingly attractive to investors and speculators. Investment resources are formed and distributed in the securities market. The issuance and circulation of securities generate various investment processes aimed at generating profits. Many investors are drawn to the high returns offered by futures market instruments, but they are equally deterred by the high risk and complexity associated with investing in financial instruments.

Futures transactions serve as a valuable instrument for managing and mitigating the risks associated with price volatility in various financial markets.

Risk assessment serves as the foundation for activity in financial markets. Investors compare the expected returns of an asset with its level of risk.

The seriousness and liquidity of securities appearing in the Ukrainian stock market, the rationality of portfolio augmentation with securities for commercial banks, and the possibility of using securities as collateral for loans are all pertinent issues for Ukrainian commercial banks today. The activities of banks in security operations, while still in the developmental stage, hold promising profitability and efficiency in the long term and thus deserve attention. Success in the stock market lies in understanding the principles of its functioning.

Within the realm of options market analysis, the profound theory of options finds widespread application in both mathematical and economic domains. Immersed in the world of options, this study delves into the intricacies of pricing models and explores the nuanced interplay between financial variables. Options, regarded as instrumental entities, facilitate risk management strategies and harbor the potential for substantial profitability. Consequently, option trading, replete with meticulously crafted strategies, blossoms into a challenging and lucrative endeavor.

In this study, we embark upon a fundamental analysis of the options market, delving into diverse models while scrutinizing the valuation of options. Our journey encompasses an exploration of the underlying principles, traversing the realm of financial mathematics. We meticulously examine the properties and behavior of options, scrutinizing their sensitivity to variables such as volatility, interest rates, and underlying asset prices. By carefully assessing these factors, we calculate the fair value of options, providing insights into their dynamics and potential profitability. Through thorough analysis, we can better understand the options and make informed investment decisions.

As a classical analysis, fractional operators are powerful tools for analyzing the behavior of functions because of their non-local nature. Fractional differential equations provide a more accurate representation of phenomena in various domains. This study justifies the need to explore other derivative and integral fractional definitions. This research aims to understand and apply fractional operator theory to the problem of physics and the Hamilton-Jacobi Equation. It developed a theory and applied it to the European Call Option Model.

The core of our investigation lies in the realm of mathematical analysis, in which we seek to connect the essence of options, Hamilton–Jacobi equations, and the fractional model. Employing a comprehensive repertoire of mathematical tools, we derive equations and formulas that underpin option pricing models. We expound upon the merits and limitations of renowned models, such as the Black-Scholes model.

The most commonly used option pricing models in options market analysis include the Black-Scholes model. This model, introduced by Fischer Black and Myron Scholes, is used for pricing European-style options. This model assumes that the underlying asset follows a geometric Brownian motion and considers factors such as strike price, time to expiration, risk-free interest rate, and volatility.

The most commonly used option pricing models in option market analysis are as follows:

Black-Scholes model: This model, introduced by Fischer Black and Myron Scholes, is used for pricing European-style options.

For option pricing, several models exist for calculating the option value, including the Black-Scholes, Black, Cox, Ross, and Rubinstein binomial models.

Let us consider the first two models.

When entering a transaction, the seller faces greater risk than the buyer. Therefore, the seller receives a certain amount, paid upfront (compensation for risk), called the option premium or its value. Using a normal distribution of prices, we propose the following formula for calculating the theoretical value of a call option (according to the Black-Scholes model):

Call:

$S\,N(d_1) - X\,e^{-RT}\,N(d\_2)$,

$d_{1,2} = \frac{ln\frac{S}{X} \pm \left(R + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$,

$R = \frac{f(d_1)}{X\,\sigma\sqrt{T}}$,

$Put = S\,N(d_1) - X\,e^{-RT}\,N(d_2)$

$-S + Xe^{-rT}$.

In the context provided, S represents the underlying asset price.

In a more concise manner, the Black-Scholes model has an advantage over the binomial model as it provides a "visible" option price and analytical expressions for all the necessary "Greeks." The deltas of call and put options are associated with the normal distribution function, and the gamma is the same for both call and put options, represented by the density of the distribution function.

Using the Black-Scholes model, we can determine the values of Θ, $Vega(V)$, and P indicators.

$\Delta\_c = N(d1),\ \Delta_p = -N(-d1)$,

$V = S\sqrt{T}\,f(d_1)$,

$Pc = T\,X\,e^{-RT}N(d_2)$,

$Pp = -T\,X\,e^{-RT}N(-d_2)$,

$\theta_c = -\frac{S\,\sigma\,f(d_1)}{2\sqrt{T}} + R\,X\,e^{-RT}N(d_2)$,

$\theta_p = -\frac{S\,\sigma\,f(d_1)}{2\sqrt{T}} + R\,X\,e^{-RT}N(-d_2)$.

The Black-Scholes model is primarily used to value European options with stocks as the underlying asset, whereas the Black model is specifically designed for pricing options on futures.

To establish a link between the two models, one can simply substitute the spot price (S) in the Black-Scholes model with the futures price (F) and consider the relationship between the futures price and the spot price, which is determined by the forward price formula.

$F = S\,e^{RT}$.

For call:

$C = e^{-RT}\left(FN(d_1) - X\,N(d\_2)\right)$.

For put option:

$P = Call - e^{-RT}\,(F - X)$,

where

$d_{1,2} = \frac{ln\frac{F}{X} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

The differential equation that models the value of a European Call option, known as Black-Scholes, has the form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{S^2\,\partial^2 V}{\partial S^2} + r\frac{S\,\partial V}{\partial S} - rV = 0 \qquad (1)$$

with the initial condition:

$V\,(S, T) = \max\{S - E, 0\}$,

$S \in (0, \infty),\ t = 0$,

where $V = f(S,t)$ is the value of a call option, S represents the price of the underlying asset, r is the risk-free rate, σ is the volatility of the asset price, E is the strike price, and T is the option 's maturity time. Its solution is classical and involves a change in variables using the heat equation.

The distinctiveness and allure of employing options stem from their versatility as a financial instrument, enabling the realization of profits in both bullish and bearish markets. Consequently, option operations yield substantial profitability. In comparison to stock investments, options consistently offer higher returns.

Presently, there is a growing focus on what is commonly referred to as structured products, specifically within the narrower subset of the expansive global market: trading in pre-designed option strategies. This trading domain exhibits notable growth and profitability.

1.1. Physics and the Hamilton–Jacobi Equation

In the realm of physics, the Hamilton-Jacobi equation plays a crucial role in describing the dynamics of classical and relativistic systems. It is a nonlinear partial differential equation closely associated with variational calculus.

The following expression gives the first-order Hamilton-Jacobi differential equation:

$$U_t + H(\nabla U) = 0,$$

where U represents the action of the system, t denotes time, H denotes the Hamiltonian, and ∇U represents the gradient of U with respect to the spatial variables.

By employing analysis and carefully examining the expressions presented in equations (1) and (2), we can explore the spatial variables, domain, initial conditions, space, energy relationships, and characteristics of the dynamic systems being modeled. In addition, we can study their solutions and the possibility of representing them in terms of an associated potential. These investigations have led to valuable insights and identified commonalities between different models that operate in diverse scenarios, such as financial and physical systems. Notably, it has been demonstrated that the Black-Scholes model involves the presence of potential and energy. This finding builds upon the research presented in [10].

It is worth emphasizing that an alternative approach, known as Econophysics, has also independently concluded the presence of a potential in the Black-Scholes model, as evidenced by recent studies documented in [16, 17]. In Econophysics, the laws and concepts of physics are applied to model economic and financial systems using mathematical tools.

Mathematical tools and concepts play a crucial role in understanding and analyzing complex physical systems across various fields of study. These tools provide a framework for modeling and investigating the behavior of systems, enabling scientists to make predictions and gain insights into the underlying principles governing their dynamics.

In (2),

$U: R^n \times (0, +\infty) \to R$, $U = f(x, t)$.

The Hamiltonian $H: R^n \to R$.

Joseph Louis Lagrange (1736-1813) made significant contributions to the field of mechanics through his formulation of Lagrange's equations. These equations, also known as the equations of motion, provide a powerful tool for describing the behavior of dynamic systems over time. They considered the number of degrees of freedom of a mechanical system and the coordinates that characterized its motion. Lagrange's equations revolutionized the field by providing a concise and elegant mathematical framework for analyzing and solving complex mechanical problems.

Building upon Lagrange's work, William R. Hamilton (1805-1865) introduced the principles of variational calculus to develop the Hamiltonian formulation of classical mechanics. Hamilton's approach represents a primary advancement in theoretical mechanics, providing an intrinsic and unified framework for describing the behavior of physical systems. The Hamiltonian formulation treated generalized coordinates and their associated momenta as independent variables, allowing for a more comprehensive understanding of the system's dynamics.

The combination of Lagrange's and Hamilton's theories laid the foundation for developing the Hamilton-Jacobi equation. This partial differential equation, named after Hamilton and Gustav Jacobi, plays a fundamental role in modern physics and has applications beyond classical mechanics. The Hamilton-Jacobi equation provides a powerful mathematical tool for solving various problems, including those related to wave propagation, quantum mechanics, and optimal control theory.

In recent years, the application of the Hamilton–Jacobi theory has expanded to address contemporary challenges in various scientific disciplines. Its relevance extends beyond traditional mechanics and applies in fields such as quantum field theory, quantum information theory, and even economic and financial modeling. Researchers have leveraged the Hamilton-Jacobi equation and its associated techniques to study complex systems, optimize resource allocation, and analyze dynamic processes in various domains.

2. Theoretical Framework

The field of generalized calculus originated from the letter exchange between W. Leibniz and G. de L'Hôpital in 1695. Since then, numerous mathematicians have contributed to its development and formalization. The non-local nature of fractional operators motivates fractional calculus, making them powerful tools for analyzing functions and accurately describing phenomena. This work focuses on the importance of fractional operators and their applications to models (1) and (2). The study incorporates Laplace, Fourier, and Mellin transforms to provide solutions to fractional differential equations. This study establishes fractional analysis as a generalization of classical analysis, comprehends integral transform theory, and tackles challenges posed by fractional equations. This study showcases the relevance and current significance of integral transforms in fractional calculus, contributing to this active research area.

2.1. The Mittag-Leffler Function

Mittag-Leffler [15] introduced the one-parameter generalization of the exponential function as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

Agarwal, who introduced the two-parameter Mittag-Leffler function, further generalized it:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \alpha)};\ \alpha, \beta > 0.$$

2.2. The Riemann-Liouville Fractional Derivative and Integral

The Riemann-Liouville fractional derivative of order α of a function $f(t)$ is defined as

$$D_z^{\alpha} f(t) = \frac{d^n}{dz^n}\left[D_z^{-(n-\alpha)} f(t)\right],$$

where n is a positive integer, $n - 1 < \alpha \leq n$ and

$$D_z^{-(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^z (x-s)^{n-\alpha-1} f(s)\, ds$$

is the Riemann-Liouville fractional integral of $f(t)$ of order $n - \alpha$.

2.3. The Mellin Integral Transform

The Mellin transform converts a function defined on positive real numbers to a new function defined on the complex plane. It is defined as follows:

For a function $f(t)$ defined on positive real numbers, the Mellin transform $F(s)$ is:

$$M(f(z); z) = f^*(z) = \int_0^{\infty} t^{s-1} f(t) dt,$$

$\gamma_1 < Re(z) < \gamma_2$.

The inverse Mellin transform allows us to recover the original function from the Mellin transform given by:

$$M^{-1}(f^*(z); t) = f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-z} f^*(z)\, dz,$$

where $t > 0, \gamma = \Re ez, \gamma_1 < \gamma < \gamma\_2$.

We restrict this investigation to the shifting and convolution properties of this transform:

$$M\,\{t^a\, f(t);\, z\} = f^*(z + a),$$

$$M\left\{\, t^n\, f^{(n)}(t);\, z\right\} = (-1)^n \frac{\Gamma(z+n)}{\Gamma(z)} f^*(z),$$

$$M\,\left\{\int_0^{\infty} f(\xi) g\left(\frac{t}{\xi}\right)\, d\xi\; \xi\; ;\, z\right\} = f^*\,(z) g^*(z).$$

3. Results

3.1. Solution of the Hamilton-Jacobi Equation

Let $L: R\hat{\ }n \times R\hat{\ }n \rightarrow R$ be a smooth function, the Lagrangian. Fixing two points x, y in R^n, for any positive t, we introduce the following functional:

$$J[r(\cdot)] = \int_0^t L\big(r(s), r(s)\big)\, ds,$$

where $r(\cdot) = \{r(\cdot),\, r_2(\cdot), \cdots, r(\cdot)\}$, belonging to the admissible class

$A = \{r(\cdot) \in C^2([0,t];\, R^n) \mid r(0) = y,\, r(t) = x\}$.

H and L are dual and convex functions, where the relation between them is (Evans L.C.) [4, p. 122]:

$H = pq - L$.

Here, pq represents a potential form associated with momentum. The following expression gives the solution to the initial value problem (2) in variational terms:

$$U(x,t) = inf\left(\int_0^t L(\, r(s))ds + g(y) \mid \dot{r}(o = y, r(t) = x\,\right), \quad (2)$$

where the infum takes over all functions $w(\cdot) \in C^1$ with $r(t) = x$. It is also required that $h: R^n \rightarrow R$ is Lipschitz continuous:

$$\sup_{\{x,y\}x \neq y} \left\{\frac{|g(x) - g(y)|}{|x - y|}\right\} < \infty.$$

Expression (2) can be minimized (Hopf-Lax formula) for $x \in R^n$, $t > 0$, in such a way that the solution takes the following form:

$$U(x,t) \leq \min_{\{y \in R^n\}}\{\, r\, L\left(\frac{x-y}{t}\right) + h(y)\} \qquad (3)$$

The latter is the weak solution to the initial value problem (2) for the Hamilton-Jacobi partial differential equation, as described in [4].

The Hamilton-Jacobi equation governs a Cauchy problem for dynamical systems, which is characterized by energy in various forms. The equation's weak solution exists in a Banach space, and the Lagrangian exhibit invariance under transformations. The Hamiltonian must be at least of class C^2, and the equation itself represents an evolution equation.

Relationships between the first-order Hamilton-Jacobi differential equation and the Black-Scholes differential equation are as follows.

The Black-Scholes differential equation, used for the valuation of a Call option, along with its respective solution, exhibits the establishing relationships with the Hamilton-Jacobi equation, where its solution takes the form $V = f(S, t)$.

The following are some relationships between equations (1) and (2) from [7]:

1. By expressing differential equation (1) in the form given by (2), we obtain

$$-\frac{\partial V}{\partial t} = \left[\frac{1}{2}\sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + r_f S\, \frac{\partial V}{\partial S} - rV\right] \qquad (4)$$

From here, we have

$$H(\nabla V) = \frac{1}{2}\sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + r_f\, S\, \frac{\partial V}{\partial S} - rV \qquad (5)$$

$H(\nabla V)$ represents the Hamiltonian associated with the Black-Scholes differential equation. Combining (4) and (5), we obtain

$V_t + H(\nabla V) = 0$.

Taking into account the initial condition $V = h$ in $R \times (0, \infty)$, $t = 0$.

2. The initial condition for V is:

$V(S, T) = \max\{S - E, 0\}$.

3. Considering (2) and (1), where S represents the price of a stock or asset in Black-Scholes, it plays the role of the spatial variable in Hamilton-Jacobi.

4. The Black-Scholes equation can be regarded as an evolution equation compared with (2).

5. The Black-Scholes model is a dynamic system because its solution provides a function for the option valuation that depends on changes over time. Therefore, the behavior of the options represents a physical field.

6. As the Black-Scholes model is a dynamic system, it can be stated that it has an associated energy, which can be reflected in the Hamiltonian as well as in its Lagrangian because both functions are convex duals. Taking reference from [11] and considering expression (3), we have that:

$$H = P_i \dot{x}^i - L\left(t, x^j, \dot{x}^j\right),$$

where P_i represents the generalized momentum. By considering $P_i\,\dot{x}^j = \Phi$ as a potential function, it can be concluded that $L = \Phi - \mathrm{H}$, therefore, the Lagrangian takes the following form:

$$L\,(S, \dot{S}, t) = \Phi - \frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} - r_f S\,\frac{\partial V}{\partial S} + rV \quad (6)$$

With $\Phi = P\,\dot{S}$, applying (3), a weak solution to this problem can be obtained using the principle of variational calculus:

$$V\,(S,t) \le \min_{y \in V}\left\{ tL\frac{S-y}{t} + g(y)\right\},$$

where $S, y \in V$. Note that $\dot{S}$ represents the variation of the stock price with respect to time.

7. The Hamilton-Jacobi equation possesses properties from the perspective of Lie symmetry analysis, using superposition, which is a powerful tool for working with the Black-Scholes equation through the theory of operators. Although it may be beyond the scope of this work, it can be demonstrated that Black-Scholes encompasses a differential operator.

8. The solutions of the Black-Scholes differential equation are defined in Banach space V defined by the following metric:

$$\left||V|\right| = D(V_1, V_2) = |V_1 - V_2|,$$

where V_1 and V_2 are solutions of (1) for t_1 and t_2, respectively. The following expression gives the solution to (4):

$$V\,(S,t) = SN(d_1) - Ee^{-r_f(T-t)}N(d_2) \qquad (7)$$

where $N(d)$, d_1, and d_2 are defined as follows:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{r^2}{2}}\,dw$$

$$d_1 = \frac{\ln SE^{-1} + \left[r_f + \frac{\sigma^2}{2}\right](T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)},$$

If V_1 is given for $t = t_1$ and V_2 is given for $t = t_2$, then

$$V_1(S,t) = SN(d_{11}) - Ee^{-r_f(T-t_1)}N(d_{12}),$$

$$V_2(S,t) = SN(d_{21}) - Ee^{-r_f(T-t_2)}N(d_{22}),$$

where

$$d_{11} = \frac{\ln SE^{-1} + \left[r_f + \frac{\sigma^2}{2}\right](T-t_1)}{\sigma\sqrt{T-t_1}},$$

$$d_{12} = d_{11} - \sigma\sqrt{T-t_1},$$

$$d_{21} = \frac{\ln SE^{-1} + \left[r_f + \frac{\sigma^2}{2}\right](T-t_2)}{\sigma\sqrt{T-t_2}},$$

$$d_{22} = d_{21} - \sigma\sqrt{T-t_2}.$$

From these expressions we have

$$|\,\Delta V| \le \frac{S}{\sqrt{2\pi}}\left|\int_{d_{11}}^{d_{21}} e^{-\frac{r^2}{2}}\,dr\right| + \frac{Ee^{-r_fT}}{\sqrt{2\pi}}\left|e_2^{-r_ft_2}\int_{-\infty}^{d_{22}} e^{-\frac{r^2}{2}}\,dr\right| + \frac{Ee^{-r_fT}}{\sqrt{2\pi}}\left|e_2^{-rt_1}\int_{-\infty}^{d_{12}} e^{-\frac{r^2}{2}}\,dr\right|.$$

Next, we verify that the initial condition of equation (1) satisfies the Lipschitz condition

$$|\,V\,(x,t) - V\,(y,t)| \le K\,|\,x-y|.$$

Let us analyze equation (27) with S > 0:

$$V\,(S,T) = \begin{cases} S-E, S \ge E \\ 0, \quad other\ case \end{cases},$$

Then

$$|\,V(x,T) - V(y,T)| = \begin{cases} |\,x-y|, x-y \ge E \\ 0, \quad other\ case \end{cases},$$

i.e., $K = 1$.

Assuming that $|V\,(x,T) - V| = |x - y|$, comparing with (7), and considering only the case of equality, we can conclude that K = 1, which satisfies the Lipschitz condition. Therefore, $V\,(S,T)$ is a Lipschitz function.

3.2. European Call Model in a Fractional Way

Problem associated with equation (1) for a European put option $E(S,t)$ with a strike price K, considers the boundary condition.

The Mellin transform and its inverse can be derived as alternative representations of the Fourier transform and its inverse. These transformations provide a different perspective on the analysis of functions and their frequency components:

$$F\,\{g(\rho); \tau\} = G(\tau) = \int_{-\infty}^{\infty} e^{i\tau\rho} g(\rho)d\rho,$$

$$F^{-1}\,\{G(\tau); \rho\} = g(\rho) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\rho} G(\tau)d\tau.$$

With the change of $e^{\rho} = t$, $i\tau = const - z$, can obtain the Mellin transform of function $f(t)$ and its inverse.

The boundary value problem for (1) for given $E(S,t)$ with exercise price K, subject to the following condition:

$$\begin{cases} E_{put}(S,T) = \theta(S) = (K-S)^+ \\ \quad E_{put}\,(0,t) = K\,e^{r(T-t)} \end{cases},$$

has one solution of the form:

$$E_{put}(S,t) = \frac{S^{\frac{1}{2}-\frac{r_f}{\sigma^2}}\,e^{-\frac{1}{8}(T-t)\left(\sigma+\frac{2r}{\sigma}\right)^2}}{\sigma\sqrt{2\pi\,(T-t)}}\int_0^K \frac{K_1}{K_2}du.$$

where $K_1 = (K-u)u^{(r/\sigma^2 - 3/2)}$,

$$K_2 = e^{\frac{1}{2}\left(\frac{ln\frac{S}{u}}{\sigma\sqrt{T-t}}\right)}$$

Note that equation (1) can be analyzed using the fractional Mellin transform.

From the definition of the Liouville fractional derivative, we found

$$(I_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s)}{(s-t)^{1-\alpha}}\, ds,$$
$$(D^\alpha y)\,(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^\infty \frac{f(s)}{(s-t)^{\alpha-n+1}}\, ds.$$

This new definition of the fractional integral and derivative, known as the right-sided Liouville fractional integral and derivative on the positive real axis, establishes a close connection with the Mellin transform. It becomes practical to directly compute the Mellin transform associated with the fractional integral and derivative of order α by using the properties of the defined operators. By establishing certain conditions, the existence of the Mellin transform can be guaranteed, allowing for a straightforward application of this transform in the analysis of fractional calculus problems.

If we consider the fractional differential equation for any value of k in the natural numbers, $k, \alpha > 0$, we have:

$$\sum_{k=0}^{n} B_k t^{\alpha+k} \left(D^{\alpha+k}\, y\right)(t) = g(t),$$

for $k = 0$, we have that:

$$\begin{cases} D_t^\alpha[f(x,t)] = C_1 g(x,t) + C_2 x^\alpha D_x^\alpha[g(x,t)] \\ \qquad + C_3 x^{\alpha+1} D_x^{\alpha+1}[g(x,t)] \\ g(x,t_0) = f(x) \end{cases} \tag{8}$$

This last expression corresponds to an ordinary fractional differential equation, as the resulting derivative acts on the variable t.

To proceed with finding the solution associated with problem (1), we consider x as the main variable on which we want to apply the Fractional Mellin Transform. Then:

$$M\left[D_t^\alpha[f(x,t)]\right] = aM[f(x,t)] + bM\left[x^\alpha\, D_x^\alpha\, [f(x,t)]\right] + cM\, x^{\alpha+1} D_x^{\alpha+1}[f(x,t)],$$

$$D_t^\alpha[F(m,t)] = aF(m,t) + b\frac{\Gamma(m+\alpha)}{\Gamma(m)}\, F(m,t) + c\frac{\Gamma(m+\alpha+1)}{\Gamma(m)} F(m,t),$$

$$D_t^\alpha[F(m,t)] = K(m)F(m,t),$$

with

$$K(m) = a + b\frac{\Gamma(m+\alpha)}{\Gamma(m)} + c\frac{\Gamma(m+\alpha+1)}{\Gamma(m)}.$$

Then for $0 < t < T$ we have:

$$D_t^\alpha[F(m,t)] = kF(m,t),$$
$$F(m,t) = r(m) \sum_{n=0}^{\infty} \frac{(k(t-T)^\alpha)^n}{\Gamma(\alpha n+1)}.$$

Now, we proceed to find the Mellin transform of the initial condition:

$$M[f(x,t_0)] = M[g(x)],$$
$$F(m,t_0) = G(m) = r(m).$$

Substituting the value of $G(m)$ in the expression of $F(m,t)$, considering the linearity of the Mellin transform and applying its inverse, we have:

$$f(x,t) = \frac{1}{2\pi i} \sum_{o}^{\infty} \frac{(t-T)^{\alpha n}}{\Gamma(\alpha n+1)} \int_{-\infty}^{\infty} x^{-m} G(m) k^n dm.$$

Note that equation (1) is a particular case of (8), i.e.:

$$\begin{cases} D_t^\alpha[V] = r_f V - r_f S_t^\alpha D_S^\alpha[V] - r_f\, S_t^{\alpha+1} D_S^{\alpha+1}[V] \\ V(S_t, T) = f(S_t) \end{cases}.$$

Considering the result obtained in (8), the Mellin transform of this equation is:

$$D_t^\alpha[F(m,t)] = \left(r_f - r_f \frac{\Gamma(m+\alpha)}{\Gamma(m)} - r_f \frac{\Gamma(m+\alpha+1)}{\Gamma(m)}\right) F(m,t).$$

Making the change of the variable:

$$K(m) = r_f - r_f \frac{\Gamma(m+\alpha)}{\Gamma(m)} - r_f \frac{\Gamma(m+\alpha+1)}{\Gamma(m)},$$

we obtain:

$$D_t^\alpha[F(m,t)] = K(m)F(m,t).$$

This is a fractional differential equation (FDO) that depends on t. In this sense, m, which represents the independent variable in the Mellin domain, is a constant. Therefore, the previous differential equation and its solution are as follows:

$$V(m,t) = r(m) \sum_{n=0}^{\infty} \frac{(k(t-T)^\alpha)^n}{\Gamma(\alpha n+1)}.$$

Considering the initial condition, passing from the domain of S_t to the domain of m to establish the Mellin transform of the initial condition and applying the inverse transform, we find that the general solution of problem (24) is of the form:

$$V\,(S_t,t) = \frac{1}{2\pi i} \int_{\infty}^{\infty} S^{-mt}\, F(m) \sum_{n=0}^{\infty} \frac{(k(t-T)^\alpha)^n}{\Gamma(\alpha n+1)} dm.$$

The Mellin operator is linear with respect to the series; therefore, the solution is:

$$V\,(S_t\,,t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(t-T)^{\alpha n}}{\Gamma(\alpha n+1)} \int_{-\infty}^{\infty} S_t^{-m} F(m) k^n\, dm.$$

This expression for $V(S_t\,,t)$ corresponds to the solution of problem (24).

4. Conclusion

The theory of options is widely used in securities transactions and has significant relevance in economics because numerous business processes can be conceptualized as options.

This model has demonstrated the connection between physics and finance, seen as a dynamic system through its relationships with the Hamilton-Jacobi equation. The European Call Option model can be likened to a system with an associated energy analogous to the concept of a Hamiltonian. The above opens up possibilities for future studies to explore the Lagrangian and potential functions within the Black-Scholes model, as shown in equation (6), where P represents the generalized momentum.

In problems involving differential equations with integer initial conditions, it is always necessary to work with the Laplace transform from the fractional calculus – Caputo´s derivative. However, in cases without integer initial conditions, it is recommended to use the Laplace transform of the fractional derivative of Riemann-Liouville, specifically for polynomial and trigonometric functions because this definition considerably inherits all the properties resulting from classical calculus.

It is of utmost importance to understand that this new perspective on calculus, initiated by Isaac Newton and Gottfried Leibniz, deviates from the traditional conception and explores new horizons, breaking paradigms, and considering possibilities that may not necessarily fall under the classical parameters. It goes beyond the conventional and demands a redefinition of classical analysis.

Investigating the effectiveness of mathematical constructions in terms of integro-differential operators will be crucial for enhancing the development of this line of research. Understanding the behavior of functions obtained under arbitrary parameters and how they can contribute to a broader interpretation of financial phenomena in the light of classical calculus will continue to be the key focus of these investigations.

For algebraic purposes and inverse transformations such as Laplace and Fourier, it is recommended to use the Mittag-Leffler function, which is of great utility when solving problems that are initially highly complex.

References

[1] DAMODARAN A. *Investment Valuation: Tools and Techniques for Determining the Value of Any Asset*. 3rd ed. Wiley, 2002.

[2] ARNOLD V.I. *Mathematical Methods of Classical Mechanics*. 2nd ed. Springer. 1989.

[3] BAAQUIE B.E., CORIANO S., and SRIKANT M. Hamiltonian and potentials in derivative pricing models: exact results and lattice simulations. *Physica A: Statistical Mechanics and its Applications*, 2004, 334(3-4): 531-557.

[4] BLACK F., and SCHOLES M. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 1973, 81(3), 637-654.

[5] DEVILLE R. On the range of the derivative of a smooth function and applications. *Journal of the Royal Academy of Exact, Physical and Natural Sciences. Series A. Mathematics*, 2006, 100(1-2): 63-74.

[6] GONZALEZ J.R., PLAZA L.F., and VASYUNKINA O. From Black-Scholes to Hamilton-Jacobi. *Contemporary Engineering Sciences*, 2018, 11(90): 4455-4463.

[7] EL HADDAD E.M. Viscosity solutions of Hamilton-Jacobi equations in infinite dimension. Stationary case. *Mathematical Publications of the Department of Mathematics of the Autonomous University of Barcelona*, 1995, 39: 173-185.

[8] HULL J.C. *Options, Futures and Other Derivatives*. 6th ed. Prentice Hall, 2006.

[9] IBRAGIMOV N.H., and GAZIZOV R.K. Lie Symmetry analysis of differential equations in finance. *Nonlinear Dynamics*, 1998, 17: 387-407.

[10] PLAZA L.F. *Analytical-numerical study of the differential equation underlying the Black-Scholes model*. Master's thesis. Technological University of Pereira, 2009.

[11] RUND H. *The Hamilton-Jacobi theory in the calculus of variations: Its role in mathematics and physics*. D. Van Nostrand Company Ltd., 1966.

[12] ZHANG H., LIU F., TURNER I., and YANG Q. Numerical solution of the time fractional Black-Scholes model governing European options. *Computers & Mathematics with Applications*, 2016, 71: 1772-1783.

[13] PODLUBNY I. *Fractional Differential Equations*. Academic Press, 1999.

[14] SAADATMANDI A., and DEHGHAN M.A. New operational matrix for solving fractional-order differential equations. *Computers & Mathematics with Applications*, 2010, 59, 1326-1336.

[15] MITTAG-LEFFLER G.M. On the new function Eα(x). *Reports of the Academy of Sciences*, 1903, 137: 554-558.

[16] OWOYEMI A., SUMIATI I., RUSYAMAN E., and FIRMAN S. Laplace Decomposition Method for Solving Fractional Black-Scholes European Option Pricing Equation. *International Journal of Quantitative Research and Modeling*, 2020, 1(4): 194-207.

[17] EDEKI S.O., RAJARAMA J., CHAKRAVERTY S., and BALEANU D. Coupled transform method for time-space fractional Black-Scholes option pricing model. *Alexandria Engineering Journal*, 2020, 59: 3239-3246.

[18] TENORIO Q.A. *Study of integral transforms as a method for solving fractional differential equations and their applications*. Master's thesis. Technological University of Pereira, 2023.

参考文:

[1] DAMODARAN A. 投資估值：確定任何資產價值的工具和技術。第三版。威利，2002。

[2] ARNOLD V.I. 經典力學的數學方法。第二版。施普林格。 1989。

[3] BAAQUIE B.E.、CORIANO S. 和 SRIKANT M. 哈密頓量和衍生品定價模型的潛力：精確結果和晶格模擬。物理學A：統計力學及其應用，2004, 334(3-4): 531-557。

[4] BLACK F. 和 SCHOLES M. 期權和公司負債的定價。政治經濟學雜誌，1973，81（3），637-654。

[5] DEVILLE R. 關於光滑函數導數的範圍及其應用。英國皇家精確、物理和自然科學院雜誌。A系列。數學，2006，100(1-2)：63-74。

[6] GONZALEZ J.R.、PLAZA L.F. 和 VASYUNKINA O. 從布萊克-斯科爾斯到漢密爾頓-雅可比。當代工程科學, 2018, 11(90): 4455-4463.

[7] EL HADDAD E.M. 無限維漢密爾頓-雅可比方程的粘度解。固定箱。巴塞羅那自治大學數學係數學出版物，1995，39：173-185。

[8] HULL J.C. 期權、期貨和其他衍生品。第6版。普倫蒂斯·霍爾，2006。

[9] IBRAGIMOV N.H. 和 GAZIZOV R.K. 金融微分方程的李對稱分析。非線性動力學，1998，17：387-407。

[10] PLAZA L.F. 黑色的-斯科爾斯模型微分方程的分析數值研究。碩士論文。佩雷拉科技大學，2009。

[11] RUND H. 變分法中的漢密爾頓-雅可比理論：它在數學和物理學中的作用。範諾斯特蘭德有限公司，1966。

[12] ZHANG H., LIU F., TURNER I. 和 YANG Q. 歐式期權時間分數黑色的-斯科爾斯模型的數值解。計算機與數學及其應用，2016，71：1772-1783。

[13] PODLUBNY I. 分數階微分方程。學術出版社，1999。

[14] SAADATMANDI A. 和 DEHGHAN M.A. 用於求解分數階微分方程的新運算矩陣。計算機與數學及其應用，2010，59，1326-1336。

[15] MITTAG-LEFFLER G.M. 關於新函數Eα(x)。科學院報告，1903年，137：554-558。

[16] OWOYEMI A.、SUMIATI I.、RUSYAMAN E. 和 FIRMAN S. 求解分數黑色的-斯科爾斯歐式期權定價方程的拉普拉斯分解方法。國際定量研究與建模雜誌，2020，1(4)：194-207。

[17] EDEKI S.O.、RAJARAMA J.、CHAKRAVERTY S. 和 BALEANU D. 時空分數黑色的-斯科爾斯期權定價模型的耦合變換方法。亞歷山大工程雜誌，2020年，59：3239-3246。

[18] TENORIO Q.A. 積分變換作為求解分數階微分方程方法的研究及其應用。碩士論文。佩雷拉科技大學，2023。