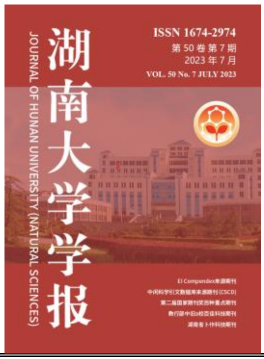

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On the Stability of Systems of Nonlinear Ordinary Differential Equations of Fractional Order

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Abstract: Ordinary differential equations of fractional order have been presented as a tool of vital importance in modeling the anomalous dynamics of various problems from the exact sciences and engineering, however, is still under discussion a clear and coherent theory analogous to the classical theory of ordinary differential equations. The nonlocal character of their fractional operators provides additional information that allows the development of a comprehensive analysis of the mathematical models. Within these fractional order differential equations, the qualitative approach is an open topic of study at present, in which stability analysis plays a preponderant role. This article presents a description of recent results on the stability of nonlinear fractional order ordinary differential equations and some analytical methods used. First of all, this article presents and describes the fundamental concepts of the study of stability of systems of ordinary differential equations of fractional order, both linear and nonlinear. The results of this research provide fundamental tools in the study of the stability of systems of nonlinear fractional order differential equations that can be applied to various models of applied sciences and engineering.

Keywords: fractional analysis, fractional differential equations, stability of fractional equations, Mittag-Leffler stability, Lyapunov functions.

分数阶非线性常微分方程组的稳定性

摘要：分数阶常微分方程已被提出作为模拟精确科学和工程中各种问题的反常动力学的至关重要的工具，然而，与经典常微分方程理论类似的清晰而连贯的理论仍在讨论中 。分数运算符的非局部特征提供了额外的信息，允许对数学模型进行全面分析。在这些分数阶微分方程中，定性方法是目前的一个开放研究课题，其中稳定性分析起着主导作用。本文介绍了非线性分数阶常微分方程稳定性的最新结果以及所使用的一些分析方法。首先，本文提出并描述了分数阶（线性和非线性）常微分方程组稳定性研究的基本概念。这项研究的结果为研究非线性分数阶微分方程系统的稳定性提供了基本工具，可应用于应用科学和工程的各种模型。

关键字：分数阶分析、分数阶微分方程、分数阶方程的稳定性、米塔格-莱弗勒稳定性、

李亚普诺夫函数。

1. Introduction

The birth of fractional calculus is as old as differential calculus, some authors establish that it originated on September 30, 1695 when L'Hopital wrote a letter to Leibniz arguing a question regarding the notation for the n −th derivative of a function $\frac{d^n}{dx^n}$, this questioning was directed on the meaning of the notation when $n = \frac{1}{2}$, to which Leibniz replied that his result it was "an apparent paradox from which interesting consequences will follow in the future" [1]. From that moment on, significant mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, Cayley, Letnikov, Laurent, Heaviside, among others, began to make contributions to the growth of fractional analysis from their lines of research. Then, in the 20th century, thanks to the developments in mathematical analysis focused on the theory of functions, new contributions were made towards the growth of this area, some focused on new formulations for the fractional derivative [2]-[4].

These advances were more noticeable when in 1974 the importance of fractional analysis was presented to the scientific community through international conferences dedicated exclusively to this area. The first took place at the University of New Haven, directed by Bertram Ross, presenting 26 articles on the subject, which served as a stimulus for numerous subsequent publications, the second conference took place in 1984 at the University of Strathclyde, in Scotland, co-directed by Adam McBride and Garry Roach, the third, directed by Katsuyuki Nishimoto, took place in 1989 at Nihon University, in Tokyo, and the fourth, co-directed by Peter Rusev, Ivan Dimovski and Virginia Kiryakova, which took place in 1996 in Varna, Bulgaria [5].

Currently it is difficult to find a field of science or engineering that does not consider concepts of fractional analysis related to the modeling of problems through fractional differential equations. From 1975 to the present, a number of articles related to the subject have been published, which undoubtedly shows its current vigor. The anomalous nature of problems from various areas of knowledge allows fractional differential equations to model their dynamics.

In addition, at present, fractional analysis has had a great development due to the applicability that it presents in various disciplines of mathematics, among them, are stochastic processes, integer differential equations, the theory of transforms, numerical analysis, the theory of viscoelasticity, in the study of the phenomenon of anomalous diffusion, in electromagnetic theory, circuit theory, biology or the physics of the atmosphere [6]-[12]. The modeling and analysis of problems applied to the above disciplines can be described using fractional type differential equations due to the non local nature of its derivative operator, which allows many of the physical phenomena that have memory and special genetic characteristics to be modeled using these equations [13]-[24].

Although fractional order differential equations have captured the attention of researchers, only a few have made significant contributions to the construction of a clear and coherent theory of these equations, in the sense analogous to the classic case of ordinary differential equations. that supports the use of this tool by applied sciences [6]. Within this study of ordinary differential equations of fractional order, the qualitative approach is an open topic of research, it has not had a significant advance that gives rise to the analysis of stability concepts on the dynamics of the solutions of said systems. There is a gap in a structured theory of stability for this type of differential equations, however, some researchers have presented results on this topic addressed from different contexts and focused on stability for linear and nonlinear fractional ordinary differential equations.

There is still no established stability theory of differential equations, however, in the last two decades valid contributions have been made towards this theory. In 1996, Matignon for the first time analyzed the stability of finite-dimensional fractional linear differential equation systems applied to control theory, for these systems presented in state space form, he presented the so-called internal and external stabilities. He mentioned that stabilities are guaranteed if the roots of some polynomial lie outside $|\arg(\sigma)| \leq \frac{\alpha\pi}{2}$, thus generalizing the known results to the ordinary case $\alpha = 1$ [25]. From that moment on, several researchers were interested in providing contributions to the study of stability of linear type fractional differential equation systems.

For the nonlinear case, the stability analysis is much more complex; however, some authors have studied the system of nonlinear differential equations of fractional order from various approaches where some stability analysis methods for the ordinary case have been extended to the fractional order. For example, in 2007 Tarasov mentions that the concept of stability with respect to fractional variations is broader than in the classical sense of Lyapunov or asymptotic stability. He states that fractional stability includes the concept of classical stability as a special case $\alpha = 1$ and also that systems can be unstable with respect to the first variation of states and be stable with respect to fractional variation. Therefore, fractional derivatives expand the possibility of investigating the properties of systems of differential equations [26].

Later, in 2009, Li, Chen and Podlubny analyzed the stability of systems of nonlinear ordinary differential equations of the fractional type using the derivative operator in the sense of Caputo or Riemann-Liouville for the order $0 < \alpha < 1$. They managed to carry out an extension of the concepts of stability of the classical order to systems of differential equations of fractional order, establishing that the drop in the generalized energy of a dynamic system does not have to be exponential for the system to be stable, this decrease in energy it can be of any type, including power law diminution.

To extend the application of fractional calculus to nonlinear systems, they proposed stability in the sense of Mittag-Leffler and Lyapunov's direct fractional method [27]. In 2010, the same authors generalized the concepts of stability in the sense of Mittag-Leffler and Lyapunov of fractional type, providing fundamental contributions for the stability of asymptotic type for the same order $0 < \alpha < 1$ [28]. Subsequently, various researchers presented stability concepts focused on Lyapunov type functions and a type of stability called strict for various derivative operators, among them is the derivative in the sense of Riemann-Liouville, Caputo, Hadamard and Caputo-Dini.

This article presents a review based on the main results that have been established on the qualitative study of ordinary differential equations of fractional order focused on the concept of stability. This review builds on the results published in [29] and supplements them to date. We apologize if we have missed some fundamental publications regarding the concepts of stability of ordinary differential equations of fractional order, however, we hope that this article will serve as a basis for future research focused on the stability of fractional equations.

This article presents three sections structured as follows: in the first section the fundamental preliminaries of fractional operators and their properties are presented and the fundamental concepts of fractional order differential equations are presented; in the second section presents the fundamental concepts of the stability of systems of nonlinear differential equations as the generalization of the direct Lyapunov method. the concepts of stability of systems of nonlinear and in the last section a discussion of the research results is made.

2. Preliminaries on Ordinary Differential Equations of Fractional Order (ODEFr)

In this section, we will initially present the fundamental preliminaries on fractional analysis and then present the relevant concepts on the study of ordinary differential equations of fractional order using derivative operators in the sense of Riemann-Liouville, Caputo, and Hadamard. The core references in this section are taken from [24], [30]-[35].

In this article, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ represents the set of complex numbers, $\mathbb{R}e(\alpha)$ represents the real part of the complex number α and $\Gamma(\cdot)$ represents the Gamma function.

Definition 1: The fractional integral in the *Riemann-Liouville* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$) of the function $x(t)$ denoted by $\mathcal{D}^{-\alpha}_{t_0,t}[x(t)]$ or $\mathcal{J}^{\alpha}_{t_0,t}[x(t)]$, is given by:

$$\mathcal{J}^{\alpha}_{t_0,t}[x(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-\alpha}}\, ds, \qquad t > t_0, (1)$$

Definition 2: The fractional derivative in the *Riemann Liouville* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by $\mathcal{D}^{\alpha}_{t_0,t}[x(t)]$, is given by:

$$\begin{aligned} \mathcal{D}^{\alpha}_{t_0,t}[x(t)] &= \frac{d^n}{dt^n}\left(\mathcal{J}^{n-\alpha}_{t_0,t}[x(t)]\right) \\ &= = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{t_0}^{t} \frac{x(s)}{(t-s)^{\alpha+1-n}}\, ds, \quad t > t_0, \end{aligned} \quad (2)$$

where $n-1 < \alpha < n$, with $n \in \mathbb{Z}_+$.

Caputo's fractional derivative allows considering initial conditions of natural order to mathematical models represented by fractional differential equations. These fractional derivative operators are defined as follows:

Definition 3: The fractional derivative in the *Caputo* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by ${}_{c}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)]$, is given by:

$$\begin{aligned} {}_{c}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)] &= \mathcal{J}^{n-\alpha}_{t_0,t}\left[x^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t} \frac{x^{(n)}(s)}{(x-s)^{\alpha+1-n}}\, ds, t > t_0, \end{aligned} \quad (3)$$

where $x^{(n)}(s)$ represents the nth derivative of the function $x(s)$ and $n-1 < \alpha < n$, with $n \in \mathbb{Z}_+$.

Another fundamental definition of the fractional operators is the fractional derivative and fractional integral in 'the sense of Hadamard.

Definition 4: The fractional integral in the *Hadamard* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$) of function $x(t)$ denoted by ${}_{\mathcal{H}}\mathfrak{D}^{-\alpha}_{t_0,t}[x(t)]$ or ${}_{\mathcal{H}}\mathcal{J}^{\alpha}_{t_0,t}[x(t)]$, is given by:

$${}_{\mathcal{H}}\mathcal{J}^{\alpha}_{t_0,t}[x(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{x(s)}{s}\, ds, t > t_0. (4)$$

Definition 5: The fractional derivative in the *Hadamard* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by ${}_{\mathcal{H}}\mathcal{D}^{\alpha}_{t_0,t}[x(t)]$, is given by:

$${}_{\mathcal{H}}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)] = \left(t\frac{d}{dt}\right)^{n} {}_{\mathcal{H}}\mathcal{J}^{n-\alpha}_{t_0,t}\left[x^{(n)}(t)\right]$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{n-\alpha+1}\frac{x(s)}{s}\,ds, t > t_0 \qquad (5)$$

where $n-1<\alpha<n$, with $n\in\mathbb{Z}_+$.

Just as in the solutions of ordinary differential equations of integer order, the exponential function plays a fundamental role, also in the solutions of differential equations of fractional order an analogous function called the Mittag-Leffler function is used.

Definition 6: The Mittag-Leffler function of a parameter α is given by:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

where $z\in\mathbb{C}$, $\mathbb{R}e(\alpha)>0$. The Mittag-Leffler function of two parameters α and β is given by:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $z,\beta\in\mathbb{C}$, $\mathbb{R}e(\alpha)>0$.

When $\beta=1$, $E_{\alpha,\beta}(z)$ coincides with the one-parameter Mittag-Leffler function, that is, $E_{\alpha,1}(z)=E_\alpha(z)$. Moreover, when $\alpha=\beta=1$, the function $E_{1,1}(z)$ is equivalent to the function e^z.

Proposition 1: If $0<\alpha<2$ and $\beta\in\mathbb{C}$ arbitrary, then for a $n\in\mathbb{Z}$ with $n\geq 1$ we have the following expansions:

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}e^{z^{\frac{1}{\alpha}}}-\sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z|\to\infty$, $|\arg(z)|\leq\alpha\frac{\pi}{2}$,

and

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z|\to\infty$, $|\arg(z)|>\alpha\frac{\pi}{2}$.

The following results can be considered generalizations of the exponential function.

Definition 7: The function

$$e_\alpha^{\lambda z}=z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^\alpha),$$

is called α −exponential function, where$z\in\mathbb{C}\backslash\{0\},\lambda\in\mathbb{C}$ and $\mathbb{R}e(\alpha)>0$.

According to proposition 1. and definition 6. it follows that $E_{1,1}(z)=E_1(z)$ and indeed $e_1^{\lambda z}=e^{\lambda z}$.

From definition 7. of α −exponential function and proposition 1. We have the following result:

Proposition 2: If $0<\alpha<2$ and $z\in\mathbb{C}$ then one has the following asymptotic equivalences for the α −exponential function:

For $|\arg(z)|\leq\alpha\frac{\pi}{2}$, you must $e_\alpha^{\lambda z}\sim\frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}e^{\lambda^{\frac{1}{\alpha}}}z$ when $|z|\to\infty$.

For $|\arg(z)|>\alpha\frac{\pi}{2}$, you must $e_\alpha^{\lambda z}\sim-\frac{1}{\lambda^2\Gamma(-\alpha)}\frac{1}{z^{\alpha+1}}$ when $|z|\to\infty$.

The following are fundamental definitions of ordinary differential equations of fractional order.

Definition 8: An ordinary differential equation of fractional order α is defined as:

$$\mathbb{D}_{t_0,t}^{\alpha}[x]=f\left(t,x,\mathbb{D}_{t_0,t}^{\alpha_1}[x],\mathbb{D}_{t_0,t}^{\alpha_2}[x],\dots,\mathbb{D}_{t_0,t}^{\alpha_{m-1}}[x]\right) \qquad (6)$$

where $x(t)$ is a real domain complex unknown function, $f(t,x,x_1,x_2,x_3,\dots,x_{n-1})$ is a known function and $\mathbb{D}_{t_0,t}^{\alpha_k}$ for $k=1,2,3,\dots,m-1$ are fractional differential operators such that $0<\mathbb{R}e(\alpha_1)<Re(\alpha_2)<Re(\alpha_3)<\cdots<Re(\alpha_{m-1})<Re(\alpha)$ for $m\in\mathbb{Z}_+, m\geq 2$.

Definition 9: An ordinary differential equation of fractional order α of Linear type is defined as:

$$\mathbb{D}_{t_0,t}^{\alpha}[x]+a_0(t)x+\sum_{k=1}^{m-1}a_k(t)\mathbb{D}_{t_0,t}^{\alpha_k}[x]=f(t) \qquad (7)$$

where $x(t)$ is a real domain complex unknown function, $a_k(t)$ for $k=0,1,2,3,\dots,m-1$ and $f(t)$ are known functions, $\mathbb{D}_{t_0,t}^{\alpha}$ for $k=1,2,3,\dots,m-1$ are fractional differential operators such that $0<\mathbb{R}e(\alpha_1)<Re(\alpha_2)<Re(\alpha_3)<\cdots<Re(\alpha_{m-1})<Re(\alpha)$ for $m\in\mathbb{Z}_+, m\geq 2$.

If the functions $a_k(t)$ are constant for all $k=0,1,2,3,\dots,m-1$, equation (7) is said to have constant coefficients, on the contrary, if at least one of them is variable the equation is said to have variable coefficients. Now, if $f(t)=0$ the linear fractional differential equation of ordinary type is said to be homogeneous.

The Cauchy type problem for the Riemann-Liouville fractional derivative presented in equation (1) is given by:

$$\begin{cases}\mathcal{D}_{t_0,t}^{\alpha}[x(t)] & =f\left(t,x,\mathcal{D}_{t_0,t}^{\alpha_1}[x],\mathcal{D}_{t_0,t}^{\alpha_2}[x],\dots,\mathcal{D}_{t_0,t}^{\alpha_{m-1}}[x]\right)\\ \mathcal{D}_{t_0,t}^{\alpha-k}[x(t)]|_{t=t_0} & =b_k,\quad b_k\in\mathbb{C},\quad k=1,2,3,\dots,m.\end{cases} \qquad (8)$$

where $m=\begin{cases}[\mathbb{R}e(\alpha)]+1 & if\ \ \alpha\notin\mathbb{Z}_+\\ \alpha & if\ \ \alpha\in\mathbb{Z}_+\end{cases}$.

The Cauchy problem for the case of the Caputo fractional derivative presented in equation (2) has the following structure:

$$\begin{cases}{}_{C}\mathfrak{D}_{a^+,t}^{\alpha}[x] & =f\left(t,x,\ {}_{C}\mathfrak{D}_{a^+,t}^{\alpha_1}[x],\ {}_{C}\mathfrak{D}_{a^+,t}^{\alpha_2}[x],\dots,\ {}_{C}\mathfrak{D}_{a^+,t}^{\alpha_{m-1}}[x]\right)\\ x^{(k)}(a) & =b_k,\quad b_k\in\mathbb{C},\quad k=1,2,3,\dots,m.\end{cases} \qquad (9)$$

where $0<\mathbb{R}e(\alpha_1)<Re(\alpha_2)<Re(\alpha_3)<\cdots<Re(\alpha_{m-1})<Re(\alpha)$ for $m\in\mathbb{Z}_+, m\geq 2$.

For this case, the initial conditions are given in terms of ordinary derivatives, which facilitates the physical interpretation.

The ordinary differential equation of fractional order presented in Equation (6) can be expressed as follows:

$$\mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x(t)]=f\left(t,x(t)\right), \qquad (10)$$

where $f:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $\tilde{\alpha}=[\alpha_1,\alpha_2,\dots,\alpha_n]^T$ with $m-1<\alpha_i<m$ for $m\in\mathbb{Z}_+$ and $i=1,2,\dots,n$, $x_k=[x_{k_1},x_{k_2},\dots,x_{k_n}]^T\in\mathbb{R}^n$ for $k=0,1,2,\dots,m-1$ represent suitable initial conditions where $x(t)=$

$[x_1(t), x_2(t) \dots, x_n(t)]^T \in \mathbb{R}^n$. The derivative operator denoted by $\mathbb{D}_{t_0,t}^{\tilde{\alpha}}$ represents the derivative in the Caputo or Riemann-Liouville sense.

If $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$ then Equation (10) can be written as:

$$\mathbb{D}_{t_0,t}^{\alpha}[x(t)] = f\big(t, x(t)\big), \qquad (11)$$

and is called a system of fractional differential equations of the same order.

Definition 8: A constant vector x_* is said to be an equilibrium point of the system of fractional differential equations (10) if and only if $f(t, x_*) = \mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x(t)]|_{x(t)=x_*}$ for all $t > t_0$.

Without loss of generality, we can consider the equilibrium point at the origin, that is, $x_* = 0$ and establish the following definition.

Definition 9: The solution $x_* = 0$ of the system of fractional order differential equations (11) is said to be:

1. Stable, if for any initial condition $x_k = [x_{k_1}, x_{k_2}, \dots, x_{k_n}]^T \in \mathbb{R}^n$ with $k = 0,1,2, \dots, m-1$, there exists $\epsilon > 0$ such that any solution $x(t)$ of equation (11) satisfies that$||x(t)|| < \epsilon$ for all $t > t_0$.
2. Asymptotically stable if it is stable and it is satisfied that $||x(t)|| \to 0$ when $t \to +\infty$.

3. The Stability of Nonlinear Fractional Ordinary Differential Equations (NFDEs)

In this section we consider the main results that have been presented on the stability of systems of ordinary differential equations of nonlinear fractional order. As we mentioned at the beginning, there is still no established theory of stability for FDEs in a similar way as for integer order ordinary differential equations; however, fundamental stability results have been presented for nonlinear FDEs because several problems from applied sciences are nonlinear.

The study on the stability of the non linear FDE is oriented in the first place to the study of the continuous dependence of the solution on the initial values. Diethem et al. have studied the system of fractional order differential equations of the same order (11) with the derivative in the Caputo sense and order $m-1 < \alpha \leq m$ applying Gronwall's inequality, we can discuss the dependence of the solution at the initial values and the structural stability [36]. The authors study the existence, uniqueness, and structural stability of solutions of nonlinear differential equations of fractional order, presenting the following results.

Theorem 1: Assume that $\Omega = [0, \chi^*] \times [x(0) - a, x(0) + a]$ with some $\chi^* > 0$ and some $a > 0$. Furthermore, let the function $f: \Omega \to \mathbb{R}$ be continuous and fulfill a Lipschitz condition with respect to the second variable. Moreover let $\alpha > 0$ such that $m - 1 < \alpha \leq m \in \mathbb{Z}_+$, and let x and y be the uniquely determined solutions of the initial value problems

$$\begin{aligned} {}_{C}\mathfrak{D}_{0,t}^{\alpha}[x(t)] &= f(t, x(t)), x_0 = x(0), \dots, x_{m-1}(0) \\ &= x^{m-1}(0) \\ {}_{C}\mathfrak{D}_{0,t}^{\alpha}[y(t)] &= f\big(t, y(t)\big), y_0 = y(0), \dots, y_{m-1}(0) \\ &= y^{m-1}(0) \end{aligned}$$

respectively. Then, there is the relation:

$$\| x - y \|_\infty = O\left(\max_{0 \leq k \leq m-1} | x_k - y_k|\right)$$

over any compact interval where both x and y exist.

According to the previous theorem, we have the structural stability of a system of nonlinear differential equations of fractional order, presented as follows.

Theorem 2: Assume that a and Ω are as in Theorem (1), and that f and $\tilde{f}$ are continuous on Ω and fulfill Lipschitz conditions with respect to the second variable. Furthermore, let x and y be the uniquely determined solutions of the initial value problems

$$\begin{aligned} {}_{C}\mathfrak{D}_{0,t}^{\alpha}[x(t)] &= f(t, x(t)), x_0 = x(0), \dots, x_{m-1}(0) \\ &= x^{m-1}(0) \\ {}_{C}\mathfrak{D}_{0,t}^{\alpha}[y(t)] &= \tilde{f}\big(t, y(t)\big), y_0 = y(0), \dots, y_{m-1}(0) \\ &= y^{m-1}(0) \end{aligned}$$

respectively. Then, we have the relation:

$$\| x - y \|_\infty = O\left(\left\| f - \tilde{f} \right\|_\infty\right)$$

over any compact interval where both x and y exist.

The conclusions of Theorems (1) and (2) can be extended to the multiple order fractional differential system (10) with the derivative in the Caputo sense.

It is noteworthy that the dependence of the solutions was studied in relation to the initial conditions of the system of fractional differential equations of the same order (11) with the derivative in the sense of Caputo, their result is as follows [37].

Theorem 3: Let the functions $f_i: \Omega \to \mathbb{R}$, for $i = 1,2, \dots, n$, where

$$\Omega = [0, \chi^*] \times \prod_{j=1}^{n} \left[x_j(0) - l_j, \quad x_j(0) + l_j\right], \quad \chi^* > 0, \quad l_j > 0, \forall j,$$

be bounded. Let $f = (f_1, \dots, f_n)$ be Lipschitz on the second variable with Lipschitz constant L. Let $x(t)$ and $y(t)$ be the solutions of the initial value problems

$$\begin{aligned} {}_{C}\mathfrak{D}_{0,t}^{\alpha}[x(t)] &= f\big(t, x(t)\big), \quad x_0 = x(0) \\ {}_{C}\mathfrak{D}_{0,t}^{\alpha}[x(t)] &= f\big(t, y(t)\big), \quad y_0 = y(0) \end{aligned}$$

respectively, where $0 < \alpha < 1$. Then,

$$\| x(t) - y(t) \|_\infty \leq \| x_0 - y_0 \|_\infty E_\alpha(Lt^\alpha),$$

where E_α is the Mittag-Leffler function.

Subsequently, the same authors generalized the previous Theorem (1) for the system of multiple order fractional differential equations (1o) with the derivative in the Caputo sense and for $m_i < \alpha_i \leq m_i + 1$, for $m_i \in \mathbb{Z}_+$, obtaining the following result [38].

Theorem 4: Let the functions $f_i: \Omega \to \mathbb{R}$, for $i = 1,2, \dots, n$, where

$$\Omega = [0,\chi^*] \times \prod_{j=1}^{n} \left[x_j(0) - l_j, \quad x_j(0) + l_j\right], \quad \chi^* > 0, \quad l_j > 0, \forall j,$$

be C^1. Let $\tilde{f} = (f_1, \ldots, f_n)$ be Lipschitz on the second variable with Lipschitz constant L. Let $\tilde{x}(t)$ and $\tilde{y}(t)$ be the solutions to the initial value problems

$$_C\mathfrak{D}^{\alpha_i}_{t_0,t}[x_i(t)] = f_i(t, \tilde{x}(t)), \quad x_i^{(k)}(0) = c_k^i, \quad 1 \le i \le n, \quad 0 \le k \le m_i$$

$$_C\mathfrak{D}^{\alpha_i}_{t_0,t}[y_i(t)] = f_i\big(t, \tilde{y}(t)\big), \quad y_i^{(k)}(0) = d_k^i, \quad 1 \le i \le n, \quad 0 \le k \le m_i$$

respectively, where $m_i < \alpha_i < m_i + 1$.

Then,

$$|x_i(t) - y_i(t)| \le \| \tilde{T} - \tilde{T}' \| E_{\alpha_i}(Lt^{\alpha_i}),$$

where $\tilde{T}(t) = [T_1, \ldots, T_n(t)]^t$, $\tilde{T}'(t) = [T'_1, \ldots, T'_n(t)]^t$, $T_i = \sum_{k=1}^{m_i} c_k^i \frac{t^k}{k!}$ and $T'_i = \sum_{k=1}^{m_i} d_k^i \frac{t^k}{k!}$, further E_{α_i} is the Mittag-Leffler function.

The stability of non linear FDEs in the Lyapunov sense was studies in [39]. The authors conducted their research for integral-differential equations of the form:

$$\begin{aligned} \mathcal{D}^{\alpha}_{t_0,t}[x(t)] &= f\big(t, x(t)\big) + \int_{t_0}^{t} K\big(t, s, x(s)\big) ds, \\ \mathcal{D}^{\alpha-1}_{t_0,t}[x(t)]|_{t=t_0} &= x_0, \end{aligned} \tag{12}$$

where $J = [t_0, t_0 + a]$, $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$, and $K \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n]$, x_0 is a real constant and $0 < \alpha \le 1$.

Applying Gronwall's lemma and Schwartz inequality, the following stability and asymptotic stability in the sense of Lyapunov were introduced.

Theorem 5: Let the function f satisfy the inequality $\| f(t, x(t)) \| \le \gamma(t) \| x \|$, and let K satisfy

$$\left\| \int_s^t K\big(\sigma, s, x(s)\big) d\sigma \right\| \le \delta(t) \| x \|, \quad s \in [t_0, t],$$

where $\gamma(t)$ and $\delta(t)$ are continuous non-negative functions such as

$$\sup\left\{ \int_{t_0}^{t} (t-s)^{\alpha-1}[\gamma(s) + \delta(s)] ds \right\} < \infty.$$

Then every solution $x(t)$ of (12) satisfies

$$\| x(t) \| \le \frac{\|x_0\|}{\Gamma(\alpha)} (t - t_0)^{\alpha-1} \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}[\gamma(s) + \delta(s)] ds \right\} < \infty.$$

Assume that $\gamma(t)$ and $\delta(t)$ are as in Theorem (5), if

$$\int_{t_0}^{t} (t-s)^{\alpha-1}[\gamma(s) + \delta(s)] ds = O[(t - t_0)^{\alpha-1}],$$

then the solution of system (12) is asymptotically stable.

If the conditions of the theorem (5) are further strengthened, the following stability result is obtained using an application of Schwartz's inequality.

Theorem 6: Assume that the function f is in $L^2(t_0, \infty)$ as a function of t and $K(t, s, x) = O\big[(t-s)^{\alpha-3/2}\big]$.

Then, for $0 < \alpha < \frac{1}{2}$, the zero solutions of system (12) is asymptotically stable.

3.1. Fractional Order Direct Lyapunov Method

Recall that the Russian mathematician and physicist Aleksandr Lyapunov established the so called direct Lyapunov method or also called the second Lyapunov method, which provides a way to analyze the stability of a system of ordinary differential equations of integer order without the need to explicitly solve the equations ordinary differentials. The general idea of the method is that the ODE system is stable if there are some candidate Lyapunov functions for that system. It is noteworthy that the direct Lyapunov method presents a sufficient condition to establish the stability of ODE systems, that is, the system can be stable even if no candidate Lyapunov function can be found to establish the stability of the system.

In [39]-[41] they started to study the stability of nonlinear FDE systems using Lyapunov functions for various generalizations of fractional derivative operators. In [27]-[28] they managed to extend the exponential stability that we know in the ODEs of integer order [42]-[43] to the so called Mittag-Leffler stability and with it they introduced the direct Lyapunov method of fractional order using the fractional derivative operator in the sense of Riemann-Liouville and Caputo.

For this, the authors initially presented the following definitions [27]-[28].

Definition 10: A solution $x(t)$ of the system (11) of nonlinear fractional differential equations is said to be *Mittag-Leffler stable* if it satisfies:

$$\|x(t)\| \le \{m[x(t_0)]E_{\alpha}(-\lambda(t - t_0)^{\alpha})\}^b \tag{13}$$

where E_α is the one parameter Mittag-Leffler function, t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally lipschitzian on $x \in B \in \mathbb{R}^n$ with Lipschitz constant m_0.

The same authors generalized Definition 10. as follows.

Definition 11: A solution $x(t)$ of the system (11) of nonlinear fractional differential equations is said to be *Generalized Mittag-Leffler stable* if it satisfies:

$$\|x(t)\| \le \left\{m[x(t_0)](t - t_0)^{-\gamma}E_{\alpha,1-\gamma}(-\lambda(t - t_0)^{\alpha})\right\}^b \tag{14}$$

where $E_{\alpha,1-\gamma}$ is the two parameter Mittag-Leffler function, t_0 is the initial time, $-\alpha < \gamma \le 1 - \alpha$, $\alpha \in (0,1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally lipschitzian on $x \in B \subset \mathbb{R}^n$ with Lipschitz constant m_0.

The following results follow from the above Definitions.

Remark 1: It follows from the properties of Mittag–Leffler function and completely monotonic function that

$$\mathcal{D}_{0,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = \mathbb{D}_{0,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = {}_{c}\mathfrak{D}_{a^+,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = t^{-\gamma}E_{\alpha,1-\gamma}(-\lambda t^\alpha)$$

is a completely monotonic function for $\alpha \in (0,1)$, $\lambda \geq 0$ and $\gamma \in [0,1-\alpha]$.

Remark 2: Mittag–Leffler stability and generalized Mittag–Leffler stability imply asymptotic stability.

Remark 3: Let $\lambda = 0$, it follows from equation (14) that

$$\|x(t)\| \leq \left[\frac{m(x(t_0))}{\Gamma(1-\gamma)}\right]^b (t-t_0)^{-\gamma b}$$

which implies that the power law stability is a special case of the Mittag–Leffler stability.

Next, the extension of the direct Lyapunov method to the case of fractional order systems are presented, leading to Mittag-Leffler stability [27].

Theorem 7: Let $x = 0$ be an equilibrium point of the system (10) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t,x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x, such that

$$\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2\|x\|^{ab},$$
$${}_{c}\mathfrak{D}_{0,t}^{\alpha}L(t,x(t)) \leq -\beta_3\|x\|^{ab},$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$ and $a, b, \beta_1, \beta_2, \beta_3$ are arbitrary positive constants.

If the function $L(t,x(t))$ satisfies the previous inequalities, then it is said to be a fractional Lyapunov function and in effect the equilibrium point $x = 0$ is said to be Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, $x_* = 0$ is globally Mittag-Leffler stable.

Considering Remark 2, the authors [27, 28] obtained an asymptotic stability result directly by weakening conditions in Theorem 7.

Theorem 8: Let $x = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t,x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x, such that

$$\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2\, {}_{c}\mathfrak{D}_{0,t}^{-\eta}\|x\|^{ab},$$
$${}_{c}\mathfrak{D}_{0,t}^{\alpha}L(t,x(t)) \leq -\beta_3\|x\|^{ab},$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$, $\eta > 0, \eta \neq \alpha, |\alpha - \eta| < 1$ and $a, b, \beta_1, \beta_2, \beta_3$ are positive arbitrary constants. If the function $L(t,x(t))$ satisfies the previous inequalities, then it is said to be a fractional generalized Lyapunov function and in effect the equilibrium point $x_* = 0$ is asymptotically stable.

The asymptotic stability of the system (11) can also be obtained through a particular type of functions, as presented below.

Definition 12: A continuous function $\alpha: [0,a) \longrightarrow [0,\infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Theorem 9: Let $x = 0$ be an equilibrium point of the system (11). If they exist, a Lyapunov function $L(t,x(t))$ and class functions $\mathcal{K}$, α_1, α_2 and α_3 that satisfy

$$\alpha_1\|x\| \leq L(t,x(t)) \leq \alpha_2\|x\|,$$
$${}_{c}\mathfrak{D}_{0,t}^{\beta}L(t,x(t)) \leq -\alpha_3\|x\|,$$

where $\beta \in (0,1)$. Then, the equilibrium point of system (11) is asymptotically stable.

The respective proofs to the above theorems can be consulted in [27]-[28].

Also for the case where $\alpha = 1$, the above results can be displayed in [42]. It is noteworthy that [44] and [45] provide comments on the results presented in the proofs of Theorems (7) and (8).

Some sufficient conditions for the local asymptotic stability of nonlinear FDEs were also derived [46], where the same order fractional differential system (11) with the Caputo derivative was considered. Then, the following stability result was given:

Theorem 10: Assume $f'(x) \in C[t_0,\infty)$ and $\alpha \in (0,1)$, then the equilibrium $x_* = 0$ of autonomous nonlinear system (11) with Caputo derivative and initial value $x(t_0) = x_0$ is locally asymptotically stable if $\lambda = f'(0) < 0$.

In [49] the authors establish a sufficient condition on the asymptotic stability of a system of nonlinear FDEs with the Caputo derivative operator, by using a Lyapunov like function. It is noteworthy that the authors used the following lemma in the presented results [47, 48].

Lemma 1: Let $v, w \in C_{1-\beta}([t_0,T],\mathbb{R})$ be locally Holder continuous for an exponent $0 < \beta < v \leq 1$, $h \in C([t_0,T] \times \mathbb{R}, \mathbb{R})$ and

1. $\mathcal{D}_{t_0,t}^{\beta}[v(t)] \leq h(t,v(t))$,
2. $\mathcal{D}_{t_0,t}^{\beta}[w(t)] \leq h(t,w(t))$, $t_0 < t \leq T$,

with non strict inequalities (1) and (2), where $v_0 = \Gamma(\beta)v(t)(t-t_0)^{1-\beta}\big|_{t=t_0}$ and $w = \Gamma(\beta)w(t)(t-t_0)^{1-\beta}\big|_{t=t_0}$. Suppose further that h satisfies the standard Lipschitz condition, then, $v_0 \leq w_0$ implies $v(t) \leq w(t)$, for $t_0 < t \leq T$.

In Lemma (1), if we replace $\mathcal{D}_{t_0,t}^{\beta}[\cdot]$ with ${}_{c}\mathfrak{D}_{t_0,t}^{\beta}[\cdot]$, but the other conditions remain unchanged, then the same result holds.

The authors of [49] initially presented a new proof of Theorem (7) and based on the results presented in that proof, they presented the following theorem.

Theorem 11: Let $x = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a continuously differentiable function $L(t,x(t)): [t_0,\infty) \times D \longrightarrow \mathbb{R}_+$ and class-$\mathcal{K}$ function α satisfying

$$L\big(t,x(t)\big) \geq \alpha\|x\|,$$
$${}_{c}\mathfrak{D}_{t_0,t}^{\beta}L\big(t,x(t)\big) \leq 0,$$

where $x \in D$ and $\beta \in (0,1)$. Then $x_* = 0$ is locally stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_* = 0$ is globally stable.

It is noteworthy that in the theorem (11), the authors assumed that the stricter requirements of the similar Lyapunov function L guarantee the existence of ${}_{c}\mathfrak{D}_{t_0,t}^{\beta}L(t,x(t))$. This fact increases the difficulty of choosing the function $L(t,x(t))$. Therefore, the authors weakened the continuous differential function $L(t,x(t))$ as $L(t,x(t)) \in C_{1-\beta}([t_0,\infty)\times D,\mathbb{R}_+)$ and presented the following theorem.

Theorem 12: Let $x_* = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin and $L(t,x(t)) \in C_{1-\beta}([t_0,\infty)\times D,\mathbb{R}_+)$. Assume that there exist a class-$\mathcal{K}$ function α such that

$$L\big(t,x(t)\big) \geq \alpha\|x\|,$$
$$\mathcal{D}_{t_0,t}^{\beta}L\big(t,x(t)\big) \leq 0,$$

where $\mathcal{D}$ is a derivative of Riemann-Liouville, $t > t_0 \geq 0$, $x \in D$, and $\beta \in (0,1)$. Then $x_* = 0$ is locally asymptotically stable. If the assumptions hold globally on $\mathbb{R}^n$, $x_* = 0$ is globally asymptotically stable.

In addition, [49] also presents sufficient conditions for the generalized stability of Mittag-Leffler and the asymptotic stability of the system.

Theorem 13: Let $x_* = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin and let $L(t,x(t)) \in C_{1-\beta}([t_0,\infty)\times D,\mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume $L(t,0) = 0$

$$L\big(t,x(t)\big) \geq \alpha\|x\|^{b},$$
$$\mathcal{D}_{t_0,t}^{\beta}L\big(t,x(t)\big) \leq 0,$$

where $t > t_0 \geq 0$, $x \in D$, and $\beta \in (0,1)$, and α, b are arbitrary positive constants. Then $x_* = 0$ is generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_* = 0$ is globally generalized Mittag-Leffler stable.

Let $x_* = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin and $L(t,x(t)) \in C_{1-\beta}([t_0,\infty)\times D,\mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume

1) there exist class-$\mathcal{K}$ functions α_i for $i = 1,2,3$ having global growth momentum at the same level and satisfying

$$\alpha_1\|x\| \leq L\big(t,x(t)\big) \leq \alpha_2\|x\|,$$
$$\mathcal{D}_{t_0,t}^{\beta}L\big(t,x(t)\big) \leq \alpha_3\|x\|,$$

2) there exists $a > 0$ such that $\alpha_1(r)$ and r^a have global growth momentum at the same level, where $t > t_0 \geq 0$, $x \in D$, and $\beta \in (0,1)$.

Then $x_* = 0$ is locally generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, $x_* = 0$ is globally generalized to be Mittag-Leffler stable.

In addition, the authors present the comparison method and apply it to analyze the stability of fractional differential equation systems by fractional differential inequalities, as follows. They consider the system (11) with $f(t,0) = 0$ and the scalar fractional differential equation

$$\begin{cases} \mathcal{D}_{t_0,t}^{\beta}[u(t)] &= g(t,u), \\ u_0 &= \Gamma(\beta)u(t)(t-t_0)^{1-\beta}\big|_{t=t_0} \end{cases} \quad (15)$$

where the initial value $u_0 \in \mathbb{R}_+$, $u(t) \in C_{1-\beta}([t_0,\infty)$, $g \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is Lipschitz in u and $g(t,0) = 0$, $0 < \beta < 1$. Also, we assume that there exists a unique solution $u(t)$ for $t \geq t_0$ for system (1) with the initial value u_0.

Theorem 14: Let $x_* = 0$ be an equilibrium point of system (11) and let $D \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a Lyapunov like function $L \in C_{1-\beta}([t_0,\infty)\times D,\mathbb{R}_+)$ and a class-$\mathcal{K}$ function α such that $L(t,0) = 0$, $L(t,x(t)) \geq \alpha\|x\|$, and $L(t,x)$ satisfies the inequality

$$\mathcal{D}_{t_0,t}^{\beta}L(t,x) \leq g\big(t,L(t,x)\big), \qquad (t,x) \in [t_0,\infty)\times D.$$

Suppose further that $g(t,x)$ is non-decreasing in x for each t. Then

1) If the zero solution of (15) is stable, then the zero solution of system (11) is stable;

2) if the zero solution of (15) is asymptotically stable, then the zero solution of system (11) is asymptotically stable.

In [50] the stability of a new system of nonlinear FDEs is studied using the derivative operator in the sense of Caputo-Hadamard, whose derivative operator is defined below. The type Caputo-Hadamard fractional derivative of fractional order α for a function $g\colon [1,\infty) \to \mathbb{R}$ is defined as

$$\begin{aligned} &{}_{c}^{\mathcal{H}}\mathfrak{D}_{t_0,t}^{\alpha}[g(t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^{n}\frac{g(s)}{s}\,ds, \end{aligned} \quad (16)$$

where $n-1 < \alpha < n$, with $n = [\alpha]+1$ and $[\cdot]$ denotes the integer part of the real number $\cdot$.

The authors present a new comparison principle and establish sufficient conditions to present the Mittag-Leffler stability and the generalized Mittag-Leffler stability for the following initial value problem for the derivative operator presented in Equation (16).

$${}_{c}^{\mathcal{H}}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] = f\big(t,x(t)\big) \quad (17)$$
$$x(t_0) = x_0,$$

where $t_0 \geq 1$, $0 < \alpha < 1$, $f \in C([1,+\infty)\times D,\mathbb{R}^n)$, $f(t,0) = 0$, $D \in \mathbb{R}^n$ be a domain containing the origin.

Definition 13: A solution $x(t)$ of the system (17) of nonlinear FDE is said to be Mittag-Leffler stable if it satisfies:

$$\|x(t)\| \leq \left\{m[x(t_0)]E_{\alpha}\left(-\lambda\left(\log\frac{t}{t_0}\right)^{\alpha}\right)\right\}^{b}$$

where E_α is the one parameter Mittag-Leffler function, t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally lipschitzian on $x \in B \in \mathbb{R}^n$ with Lipschitz constant m_0.

Definition 14: A solution $x(t)$ of the system (17) of nonlinear FDE is said to be generalized Mittag-Leffler stable if it satisfies:

$$\|x(t)\| \leq \left\{ m[x(t_0)] \left(\log\frac{t}{t_0}\right)^{-\gamma} E_{\alpha,1-\gamma}\left(-\lambda\left(\log\frac{t}{t_0}\right)^{\alpha}\right)\right\}^{b}$$

where $E_{\alpha,1-\gamma}$ is the two parameter Mittag-Leffler function, t_0 is the initial time, $-\alpha < \gamma \leq 1-\alpha$, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally lipschitzian on $x \in B \subset \mathbb{R}^n$ with Lipschitz constant m_0.

The following comparison results apply to the following initial value problem.

$$ {}^{\mathcal{H}}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{t_0,t}[w(t)] = y(t, w(t)) \qquad (18)$$
$$w(t_0) = w,$$

where $y \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$ is Lipschitz in w and $y(t,0) = 0$.

Lemma 2: Let $h: [t_0, T) \rightarrow \mathbb{R}$ be a locally Hölder continuous such that for any $t_1 \in [t_0, T)$, we have $h(t_1) = 0$ and $h(t) \leq 0$ for $t_0 \leq t \leq t_1$. Then it follows that ${}_{\mathcal{H}}\mathcal{D}^{\alpha}_{t_0,t}[h(t_1)] \geq 0$ for $0 < \alpha < 1$.

The above Lemma (2) also holds for the derivative operator in the Caputo-Hadamard sense ${}^{\mathcal{H}}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{t_0,t}[h(t_1)]$.

Theorem 15: Assume that the following conditions are satisfied. Let $h: [t_0, T] \times D \rightarrow \mathbb{R}$ be a continuously differentiable function, the inequality holds

$$ {}^{\mathcal{H}}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{t_0,t}[h(t)] \leq y(t, h(t)), \qquad t \geq t_0, t_0 \geq 1,$$

$Y(t) = Y(t, t_0, w_0)$ is the maximal solution of the initial value problem equation (18) existing on $[t_0, T]$. Then we have $h(t) \leq Y(t)$, for $t \in [t_0, T]$, whenever $w_0 \geq h(t_0)$.

The following result presented in [50] generalizes the stability for the system (17) using a function similar to Lyapunov's.

Theorem 16: Assume the following conditions are satisfied. There exists a function $L(t,x(t)): [t_0,\infty) \times D \rightarrow \mathbb{R}^+$ be a continuously differentiable function and locally Lipschitz respect to x such that $L(t,0) = 0$, the inequality holds

$$ {}^{\mathcal{H}}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{t_0,t}L(t,x(t)) \leq y\left(t, L(t,x(t))\right),$$
$$(t,x) \in [t_0,\infty) \times D,$$

The maximal solution $Y(t, t_0, w_0)$ of the IVP Equation (18) exists on $[t_0, \infty)$. Then we have:

1. If there exists φ de clase-$\mathcal{K}$ such that

$$L(t,x(t)) \geq \varphi(\| x \|)$$

then the stability of the zero solution of Equation (18) implies the stability of the zero solution of Equation (17) and the asymptotic stability of the zero solution of Equation (18) implies the asymptotic stability of the zero solution of Equation (17).

2. If

$$L(t,x(t)) \geq b \| x \|^{\beta}$$

where $b > 0$ and $\beta > 0$, then the generalized Mittag-Leffler stability of the zero solution of Equation (18) implies the generalized Mittag-Leffler stability of the zero solution of Equation (17); the Mittag-Leffler stability of the zero solution of Equation (18) implies the Mittag-Leffler stability of the zero solution of Equation (17).

Also in [51] the concepts of Mittag-Leffler stability and generalized Mittag-Leffler stability are generalized for a special type of nonlinear FDE system using a derivative operator generalization in the sense of Caputo. With this generalization of the Mittag-Leffler stability, they extend their results using the generalized comparison principle and thereby establish the direct Lyapunov method for the initial value problem:

$$ {}^{\mathcal{H}}_{\mathrm{c}}\mathfrak{D}^{\alpha,\rho}_{t_0,t}[x(t)] = f(t, x(t)) \qquad (19)$$
$$x(t_0) = x_0,$$

where $0 < \alpha < 1$, $\rho > 0$, $x(t) \in \mathbb{R}^n$ and $f[t_0, +\infty \times D \rightarrow \mathbb{R}^n)$. The definition of the derivative operator ${}_{\mathrm{c}}\mathfrak{D}^{\alpha,\rho}_{t_0,t}[x(t)]$ can be consulted and studied in [51].

In [52]-[53], the authors study the stability of nonlinear FDE systems where they present a new definition of exponential stability of fractional order systems and obtain sufficient conditions for the exponential stability of FDE systems using the notion Lyapunov stability tests, based on the second Lyapunov method.

In [54]-[56] the stability of the zero solution of a nonlinear fractional differential equation is studied using a new derivative operator in the sense of Caputo-Dini. The authors establish several sufficient conditions of stability, uniform stability and asymptotic uniform stability, based on the new definition of the derivative of the Lyapunov functions and the new comparison result.

Also in [57]-[60] the practical stability results of an integer order ordinary differential equation were extended for a non linear Caputo fractional differential equation and is studied using functions similar to Lyapunov's. Also, several sufficient conditions are established for practical stability, quasi practical stability, strong practical stability of the zero solution, and the corresponding uniform types of practical stability.

In [61] and [62] the problem of stability and stabilization of nonlinear systems of fractional order for $0 < \alpha < 2$ is investigated. Based on the Lyapunov stability theorem of fractional order, the S-procedure, and the Mittag-Leffler function, the stability conditions ensure local stability and stabilization. As in [63], stability theorems for nonlinear FDE systems with Caputo and Riemann Liouville derivatives are presented and, in particular, they derive conditions for F-stability of nonlinear FDEs.

In [64], the generalized Mittag-Leffler stability of Hilfer's non-autonomous fractional system is studied using the direct Lyapunov method. A new Hilder type fractional comparison principle is also tested. Lyapunov's direct fractional method combined with the Hilfer-type fractional comparison principle is presented.

Some authors have established results for the second fractional Lyapunov method through the construction of special Lyapunov type functions, such as [65], where the authors propose a lemma for the fractional derivative operator in the Caputo sense. This result is useful for applying the fractional order extension of the direct Lyapunov method and is presented below.

Lemma 3: Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$

$$\frac{1}{2}\ {}_{c}\mathfrak{D}^{\alpha}_{t_0,t}x^2(t) \leq x(t)\,{}_{c}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)]$$

for all $\alpha \in (0,1)$.

In the case that $x(t) \in \mathbb{R}^n$, the previous Lemma (3) is still valid.

Lemma 4: That is, for all $\alpha \in (0,1)$ and for all $t \geq t_0$, we have

$$\frac{1}{2}\ {}_{c}\mathfrak{D}^{\alpha}_{t_0,t}x^T(t) \leq x^T(t)\,{}_{c}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)]$$

For the fractional order system

$${}_{c}\mathfrak{D}^{\alpha}_{t_0,t}x(t) = f\big(x(t)\big)$$

where $0 < \alpha < 1$, $x_* = 0$ is the equilibrium point and $x(t) \in \mathbb{R}$, if the following condition is satisfied $x(t)f(x(t)) \leq 0$, $\forall x$, then the origin of the system (11) is stable. And if

$$x(t)f\big(x(t)\big) < 0, \quad \forall x \neq 0$$

then the origin of the system (10) is asymptotically stable.

When the system (11) is vectorial, that is when $x(t) \in \mathbb{R}^n$, the previous Lemma 4. is still valid. The proof of it is straightforward, using a candidate Lyapunov function given by $L(x(t)) = \frac{1}{2}x^T(t)x(t)$ and applying the Lemma (3). The authors applied the previous results in the stability analysis of nonlinear FDE systems using special Lyapunov functions.

In [66]-[69], the authors generalize the quadratic forms to fractional order to obtain Lyapunov functions to establish the stability of FDE systems, they also present the converse theorems for fractional order systems of Caputo.

4. Conclusion

The study of the stability of systems of ordinary differential equations of fractional order is one of the fundamental topics within the study of various models of applied sciences and engineering. This article hopes to collect a large part of the fundamental contributions presented on the stability study from the first results presented by Matignon to the present, as we mentioned before, we apologize if some fundamental references are not found in this research. We try to present the fundamental results within the study of the stability of nonlinear FDEs.

With this study we observe that significant progress has been made in the construction of grounded stability theory for FDEs. Where it can be seen that the stability of the linear FDEs has been well investigated and for the stability of the non-linear FDEs, fundamental results have been provided such as the second fractional type Lyapunov method, the functions of fractional Lyapunov type for fractional differential systems. At present, generalizations about the concepts of stability for FDE systems continue to be presented, we hope that this research will be very useful for studies of the stability of fractional differential equations.

This work is novel because it aims to contribute to the current debate in the literature on the stability of systems of nonlinear fractional order differential equations, which are of vital importance in the modeling of various problems in applied sciences. The approach to the stability of systems of nonlinear fractional equations is a current topic of study which presents its advantages in the analysis of various problems from science and engineering.

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