第 49 卷 第 6 期
2022 年 6 月

湖南大学学报（自然科学版）
Journal of Hunan University (Natural Sciences)

Vol. 49 No. 6
June 2022

Open Access Article https://doi.org/10.55463/issn.1674-2974.49.6.13

Stochastic Optimal Control of Economic Growth Model under Research and Development Investment with Kalman Filtering Approaches

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Abstract: This article examines an economic growth model that expresses the interaction between production, technology stock, and research and development (R&D) investments. The goal of this study is to maximize production. Considering the presence of Gaussian white noises, this model is reformulated as a stochastic optimal control problem, where the R&D investment rate is defined as the control input. We aim to explore the efficiency of Kalman filtering approaches for solving this problem. Here, the extended Kalman filter (EKF) and unscented Kalman filter (UKF) are applied for state estimation. The state equation linearization is made in the EKF, while the unscented transform is taken in the UKF for generating a set of sigma points. These approaches aim to estimate the state dynamics from different perspectives. With these state estimates, two different computational algorithms are proposed, the EKF for state-control (EKF4SC) and UKF for state-control (UKF4SC) algorithms. The optimal control policy is designed to minimize the cost function. For illustration, the model's parameters are considered in the simulation experiment. The simulation results showed that the UKF has higher accuracy in the state estimation with the smallest mean squares of error compared with the EKF. Moreover, the optimal control policy based on the state estimate generated from the UKF could optimize the cost function of the problem. Hence, the results show the efficiency of the algorithms proposed. In conclusion, the application of the Kalman filtering algorithms to the economic growth model is presented. The significance of this study is to provide a stochastic optimal control model for the economic growth problem and to suggest an efficient computational approach for solving this stochastic optimal control problem.

Keywords: economic growth model, research and development investment, nonlinear stochastic optimal control, state estimation, Kalman filtering.

基于卡尔曼滤波方法的研发投资经济增长模型随机最优控制

摘要：本文研究了生產、技術存量和研發投資之間相互作用的經濟增長模型，以提高生產量。考慮了高斯白噪聲的存在，該模型重述為隨機最優控制問題，并定義研發投資率為控制輸入，以探索卡爾曼濾波方法解決此問題的效率。應用擴展卡爾曼濾波器(EKF)和無跡卡爾曼濾波器(UKF)於狀態估計的分別是狀態方程在EKF中需線性化，而在UKF中使用無跡變換生成一組采样點。這些方法從不同的角度估計狀態動態。为此提出兩種不同的算法，並設計最優控制策略以最小化成本函數。為了說明，在模擬實驗中考慮了模型的參數。結果表明，UKF在均方誤差最小的狀態估計中具有更高的精度，基於UKF狀態估計的最優控制策略可以優化問題的成本函數，顯示所提出算法的效率。結論，介紹了卡爾曼濾波算法在經濟增長模型中的應用。本研究的意義在於為經濟增長問題提供一個隨機最優控制模型，並提出一種

Received: March 1, 2022 / Revised: April 3, 2022 / Accepted: May 5, 2022 / Published: June 30, 2022

Fund Project: The Ministry of Higher Education, Malaysia (The Fundamental Research Grant Scheme (FRGS), Reference Code FRGS/1/2018/STG06/UTHM/02/5), Universiti Tun Hussein Onn Malaysia

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有效的計算法來解決這個隨機最優控制問題。

关键词：经济增长模型、研发投资、非线性随机最优控制、状态估计、卡尔曼滤波。

1. Introduction

Technological innovation is an important driving force for sustainable economic development and can be achieved through research and development (R&D) investment [1]. This is because R&D is a key determinant of long-run productivity, and the effectiveness of R&D investment will lead to sustainable economic growth [2]. Moreover, the R&D investment can increase the resource endowment of innovation activities, optimize production conditions, and promote the output of technological innovation [3]. Thus, economic growth promotes technological innovation advances and improves people's living standards.

On the other hand, optimal control theories have been widely used in economics over nearly fifty years of growth, and many economic growth models have been well utilized. In the economics literature, the issue of R&D investment was studied, and some methods have been applied to estimate the effect of R&D investment on economic growth, productivity, and output growth. As a result, a rate of return on the R&D investment is generated [4]. Despite the growing demands for more workable and reliable solutions to optimal control problems, many real-world problems are very complicated to obtain practical solutions.

Therefore, the demand for optimizing and upgrading the economic growth model is required as accurate economic growth models are expected to directly benefit the management of economic development [5]. The purpose of the economic growth model is to maximize the total production assuming that the market is stable and the demand for the product exists at all times [6]. So, the control system should be improved and promoted to accelerate the transformation of economic development and improve economic quality and efficiency [7].

From the past studies, the Kalman filter has been widely applied in economic growth, either for estimating, predicting, or forecasting purposes. Some applications of the Kalman filter approach in economic growth studies include the estimation of the potential gross domestic product (GDP) [8], Solow-Cobb-Douglas economic growth model [9], time-varying economic impacts on tourism demand [10], and balance-of-payments constraint and economic growth [11]. Hence, improving the Kalman filter and exploring economic growth are active research, which will leave a great contribution to communities and countries.

In this article, the nonlinear control model that describes the change in production and R&D investment [5] is studied, where the R&D investment rate is considered the control input. Different from the study in [5], in this paper, the stochastic optimal control problem, which is adding the random Gaussian white noises into the model, is introduced. Using the extended Kalman filter (EKF) and unscented Kalman filter (UKF) approaches [12, 13], the state dynamics are estimated. The sum of squares of the output errors is minimized, where the performances of the state estimation between the EKF and UKF approaches are compared. For simplicity, a linear feedback control law is then designed based on the state estimates to minimize the functional cost. This article's main contribution is applying both Kalman filtering approaches in solving the economic growth model's nonlinear stochastic optimal control problem.

The rest of this paper is organized as follows. In Section 2, the economic growth model is considered, and this model's stochastic optimal control problem is described. Section 3 discusses the computational approaches of the EKF and UKF used to solve the optimal control problem of economic growth with the R&D investment as the control input. Their calculation procedures are summarized as the EKF for state-control (EKF4SC) algorithm and UKF for state-control (UKF4SC) algorithm. Section 4 presents the significant results of the state estimate and feedback control policy, which are the optimal solution to the economic growth problem. Finally, some concluding remarks are made.

2. Methods

Here, the problem statement of the economic growth model is described, and the methodology of Kalman filtering approaches, which are the EKF and UKF, is discussed.

2.1. Economic Growth Model

Consider a nonlinear control system, which describes the interaction between the production, the technology stock, and R&D investments [5] as follows,

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t) + b\left(\frac{x_2(t)}{x_1(t)}\right)^{\gamma} x_1(t) - cu(t)x_1(t), \\ \dot{x}_2(t) &= du(t)x_1(t), \end{aligned} \tag{1}$$

with the initial state $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$, and $x_{0,1}, x_{0,2} > 0$, $t \in [0,T]$, where $x_1(t)$ is the production and $x_2(t)$ is the total technology stock, whereas $u(t)$ is the R&D intensity satisfying the inequality $0 < u_1 \leq$

$u(t) \le u_2$, for almost all time $t \in [0,T]$ with $u_1 > 0$ and $u_2 > u_1$ as the minimal and maximal possible R&D intensity values. Here, a is the R&D contribution to increases in production, b is the non-R&D contribution to increases in production, c is the discounted marginal productivity of technology and d is the coefficient of expenses for technology development. These parameters are positive. Meanwhile, $\gamma \in (0,1)$ is an elasticity parameter of technology stock to the production, and T is the end of the planning period.

The state equation of the nonlinear control system (1) is formulated by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} ax_1(t) + b\left(\dfrac{x_2(t)}{x_1(t)}\right)^{\gamma} x_1(t) - cu(t)x_1(t) \\ du(t)x_1(t) \end{pmatrix} \quad (2)$$

and its equivalent discrete-time model is

$$x(k+1) = f(x(k), u(k)) \quad (3)$$

where $f : \Re^2 \times \Re \to \Re^2$ represents the plant function given by

$$f(x(k),u(k)) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \tau \begin{pmatrix} ax_1(k) + b\left(\dfrac{x_2(k)}{x_1(k)}\right)^{\gamma} x_1(t) - cu(k)x_1(k) \\ du(k)x_1(k) \end{pmatrix} \quad (4)$$

with τ is the sampling time for $k = 0,1,\cdots,N$.

In the presence of process noise $\omega(k) \in \Re^2$, $k = 0,1,\cdots,N-1$, measurement noise $\eta(k) \in \Re$, $k = 0,1,\cdots,N$, and random disturbances, the model in (3) can be written as follows:

$$x(k+1) = f(x(k),u(k)) + G\omega(k) \quad (5)$$

where $x(k) = (x_1(k), x_2(k))^{\mathrm{T}} \in \Re^2$ is the state variable, and G is a 2×2 coefficient matrix. The output measurement is denoted by

$$y(k) = h(x(k)) + \eta(k) \quad (6)$$

with $h : \Re^2 \to \Re$ is the output channel function defined by

$$h(x(k)) = x_1(k). \quad (7)$$

The random disturbances $\omega(k)$ and $\eta(k)$ are Gaussian white noise sequences with zero mean, and their covariance matrices are Q_ω and R_η, respectively. The initial state $x(0) = x_0$ is a random vector, and its expected value and error covariance are given by

$$E[x_0] = \bar{x}_0 \text{ and } E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^{\mathrm{T}}] = M_0,$$

where $M_0 \in \Re^{2\times 2}$ is a positive definite matrix, and $E[\cdot]$ is the expectation operator. It is assumed that the initial state, process noise, and measurement noise are statistically independent.

Hence, the aim is to find a set of the optimal control sequences $u(k)$, $k = 0,1,\cdots,N-1$, such that the cost function

$$J(u) = E\left[\varphi(x(N)) + \sum_{k=0}^{N-1} L(x(k),u(k))\right] \quad (8)$$

is minimized over the dynamical system defined by (5) and (6). Here, $\varphi : \Re^2 \to \Re$ is the terminal cost function, and $L : \Re^2 \times \Re^2 \to \Re$ is the operating cost function.

Therefore, this problem is referred to as a discrete-time nonlinear stochastic optimal control problem for the economic growth model under R&D investment, and it is regarded as Problem (P).

2.2. Extended Kalman Filtering

Consider the state mean propagation for the state dynamics in (3),

$$\bar{x}(k+1) = f(\bar{x}(k),u(k)) \quad (9)$$

where $\bar{x}(k)$ is the state mean sequences, and define the weighted least-squares error [13, 14],

$$J_{lse}(x) = \frac{1}{2}(x(k) - \bar{x}(k))^{\mathrm{T}} M_x(k)^{-1}(x(k) - x(k)) + \frac{1}{2}(y(k) - h(x(k)))^{\mathrm{T}}(R_\eta)^{-1}(y(k) - h(x(k))). \quad (10)$$

to be minimized. By having the necessary condition $\nabla J_{lse}(x) = 0$, the optimal state estimate is obtained by

$$\hat{x}(k) = \bar{x}(k) + K_f(k)(y(k) - \bar{y}(k)) \quad (11)$$

$$\bar{x}(k+1) = f(\hat{x}(k),u(k)) \quad (12)$$

$$\bar{y}(k) = h(\bar{x}(k)) \quad (13)$$

where $\hat{x}(k)$ is the filtered state estimate, $\bar{x}(k)$ is the predicted state estimate, and $\bar{y}(k)$ is the output estimate.

Here, the Kalman filter gain is

$$K_f(k) = M_x(k)C^{\mathrm{T}} M_y(k)^{-1} \quad (14)$$

whereas the state error covariance matrices are

$$P(k) = M_x(k) - M_x(k)C^{\mathrm{T}} M_y(k)^{-1} CM_x(k) \quad (15)$$

$$M_x(k+1) = AP(k)A^{\mathrm{T}} + GQ_\omega G^{\mathrm{T}} \quad (16)$$

and the output error covariance matrix is

$$M_y(k) = CM_x(k)C^{\mathrm{T}} + R_\eta \quad (17)$$

with the initial condition $M_x(0) = M_0$. The filtered state error covariance $P(k)$, the predicted state error covariance $M_x(k)$, and the output error covariance $M_y(k)$ are positive definite matrices [15, 16]. The linearization of the dynamical system (11)-(12) will be done using the following Jacobian matrices:

$$A \approx \nabla_{x(k)} f \text{ and } C \approx \nabla_{x(k)} h.$$

Thus, it is noticed that (11) is the measurement update, and (12) is the time update. These two equations are known as the Kalman filtering equations. This method is commonly known as the extended

Kalman filtering (EKF) approach.

2.3. Unscented Kalman Filtering

Assume that the n-dimensional random state vector x has mean $\bar{x}$ and covariance P_{xx}. On this basis, a set of sigma points [12] is denoted by

$$\chi = (\chi_0, \chi_i, \chi_{i+n}) \tag{18}$$

with the components

$$\chi_0 = \bar{x}\,, \tag{19}$$

$$\chi_i = \bar{x} + (\sqrt{(n+\lambda)P_{xx}})_i\,, \tag{20}$$

$$\chi_{i+n} = \bar{x} - (\sqrt{(n+\lambda)P_{xx}})_{i+n}\,, \tag{21}$$

for $i = 1, \cdots, n,$ and the weights

$$W_0^{(m)} = \frac{\lambda}{n+\lambda}\,, \tag{22}$$

$$W_0^{(c)} = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta)\,, \tag{23}$$

$$W_i^{(m)} = W_i^{(c)} = \frac{1}{2(n+\lambda)}\,, \tag{24}$$

for $i = 1, \cdots, 2n.$ Here, $\left(\sqrt{(n+\lambda)P_{xx}}\right)_i$ is the ith column of the matrix square root $(n+\lambda)P_{xx}$, and W_i is the weight value, which satisfies the condition

$$\sum_{i=0}^{2n} W_i^{(c)} = 1 \text{ and } \sum_{i=0}^{2n} W_i^{(m)} = 1\,. \tag{25}$$

Here,

$$\lambda = \alpha^2(n+\kappa) - n \tag{26}$$

is a scaling factor, where α determines the spread of the sigma points around $\bar{x}$, and it is assigned to a small positive value (10^{-3}) in the range $0 < \alpha \le 1$. And κ is a secondary scaling factor, which is in the range $0 \le \kappa \le 3$ that is usually set to zero (0). Moreover, β is used to incorporate prior knowledge of the distribution of the state x with $\beta \ge 0$ and $\beta = 2$ is the optimal value for Gaussian distributions.

2.3.1. Unscented Transformation

According to (18), the sigma points are propagated through the nonlinear function h given in (13) to generate the transformed sigma points from

$$\Upsilon_i = h(\chi_i) \tag{27}$$

for $i = 1, \cdots, 2n.$ By using a weighted sample mean and covariance of the transformed sigma points, the mean and covariance for the output variable y are approximated from

$$\bar{y} = \sum_{i=0}^{2n} W_i^{(m)} \Upsilon_i \tag{28}$$

$$P_{yy} = \sum_{i=0}^{2n} W_i^{(c)} (\Upsilon_i - \bar{y})(\Upsilon_i - \bar{y})^{\mathrm{T}} + R_\eta \tag{29}$$

where R_η is the output noise covariance.

Notice that the unscented transformation (28) and (29) are more accurate than the linearization method for propagating means and covariances of the nonlinear function.

2.3.2. State Estimation

Consider the initial value of the predicted mean and covariance of the state,

$$\bar{x}_0 = E[x_0] \tag{30}$$

$$P_0 = E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^{\mathrm{T}}] \tag{31}$$

and the sigma points using a priori mean and covariance of the state are provided by

$$\chi(k) = \left[\bar{x}(k) \quad \bar{x}(k) + \gamma\sqrt{P(k)} \quad \bar{x}(k) - \gamma\sqrt{P(k)} \right] \tag{32}$$

where $\gamma = \sqrt{n+\lambda}$ as stated in (20) for $k = 0, 1, \cdots, N-1.$ Therefore, in the time update procedure, the state of the transformed sigma points is predicted from

$$\chi(k+1) = f(\chi(k), u(k)) \tag{33}$$

with the estimated state mean

$$\hat{x}^-(k) = \sum_{i=0}^{2n} W_i^{(m)} \chi_i(k) \tag{34}$$

and the estimated state error covariance

$$P^-(k) = \sum_{i=0}^{2n} W_i^{(c)} (\chi_i(k) - \hat{x}^-(k)) \times (\chi_i(k) - \hat{x}^-(k))^{\mathrm{T}} + Q_w \tag{35}$$

where Q_ω is the process noise covariance.

On the other hand, in the measurement update procedure, the output of the transformed sigma points is measured by

$$\Upsilon(k) = h(\chi(k)) \tag{36}$$

with the estimated observation

$$\hat{y}^-(k) = \sum_{i=0}^{2n} W_i^{(m)} \Upsilon_i(k) \tag{37}$$

and the observation error covariance

$$P_{yy}(k) = \sum_{i=0}^{2n} W_i^{(c)} (\Upsilon_i(k) - \hat{y}^-(k)) \times (\Upsilon_i(k) - \hat{y}^-(k))^{\mathrm{T}} + R_\eta \tag{38}$$

while the state estimate mean is updated by

$$\hat{x}(k) = \hat{x}^-(k) + K_f(k)(y(k) - \hat{y}^-(k)) \tag{39}$$

with the updated state error covariance

$$P(k) = P^-(k) - K_f(k) P_{yy}(k) K_f(k)^{\mathrm{T}} \tag{40}$$

where

$$K_f(k) = P_{xy}(k) P_{yy}(k)^{-1} \tag{41}$$

$$P_{xy}(k) = \sum_{i=0}^{2n} W_i^{(c)} (\chi_i(k) - \hat{x}^-(k))(\Upsilon_i(k) - \hat{y}^-(k))^{\mathrm{T}}. \tag{42}$$

Here, R_η is the observation noise covariance, $K_f(k)$ is the Kalman filter gain, and $P_{xy}(k)$ is the cross-correlation matrix. This method is known as the UKF approach [12].

2.4. Optimality Conditions

For the measurable purpose, the cost function (8) is written in its expectation form [13]

$$J(u) = \varphi(\bar{x}(N)) + \sum_{k=0}^{N-1} L(\bar{x}(k), u(k)) . \quad (43)$$

Define the Hamiltonian function

$$H(k) = L(\bar{x}(k), u(k)) + p(k+1)^T (f(\hat{x}(k), u(k)) \quad (44)$$

where $p(k)$ is a 2×1 co-state vector to be determined later. The augmented cost function is written by

$$J'(u) = \varphi(\bar{x}(N)) + p(0)^{\mathrm{T}} \bar{x}(0) - p(N)^{\mathrm{T}} \bar{x}(N) + \sum_{k=0}^{N-1} (H(k) - p(k)^{\mathrm{T}} \bar{x}(k)). \quad (45)$$

According to the Lagrange multiplier theory, at a constrained minimum, the increment dJ' should be zero [15, 16, 17]. Hence, the following necessary conditions are derived.

(a) Stationary condition

$$\nabla_u L(\bar{x}(k), u(k)) + \nabla_u f(\hat{x}(k), u(k))^{\mathrm{T}} p(k+1) = 0 \quad (46)$$

(b) State equation

$$\bar{x}(k+1) = f(\hat{x}(k), u(k)) \quad (47)$$

(c) Costate equation

$$p(k) = \nabla_x L(\bar{x}(k), u(k)) + \nabla_x f(\hat{x}(k), u(k))^{\mathrm{T}} p(k+1) \quad (48)$$

(d) Boundary conditions

$$\bar{x}(0) = \bar{x}_0 \text{ and } p(N) = \nabla_{x(N)} \varphi(\bar{x}(N)) \quad (49)$$

2.5. Optimal Control Law

Assume that the cost function (43) can be approximated into its quadratic criterion, that is,

$$\varphi(\bar{x}(N)) \approx \frac{1}{2} \bar{x}(N)^{\mathrm{T}} S(N) \bar{x}(N) \quad (50)$$

$$L(\bar{x}(k), u(k)) \approx \frac{1}{2} \left(\bar{x}(k)^{\mathrm{T}} Q \bar{x}(k) + u(k)^{\mathrm{T}} R u(k) \right) \quad (51)$$

where $S(N)$, Q and R are the weighting matrices. Hence, the optimality conditions (46) and (48) are simplified as follows,

$$Ru(k) + B^{\mathrm{T}} p(k+1) = 0 \quad (52)$$

$$p(k) = Q\bar{x}(k) + A^{\mathrm{T}} p(k+1) \quad (53)$$

with the Jacobian matrices,

$$A \approx \nabla_x f \text{ and } B \approx \nabla_u f$$

and

$$p(N) = S(N)\bar{x}(N) . \quad (54)$$

Suppose the costate equation [16] has the following solution

$$p(k) = S(k)\bar{x}(k) \quad (55)$$

and consider this solution in the optimality conditions (52) and (53). After doing some algebraic manipulations, the linear feedback control law is designed as follows,

$$u(k) = -K(k)\bar{x}(k) \quad (56)$$

with

$$K(k) = (B^{\mathrm{T}} S(k+1) B + R)^{-1} B^{\mathrm{T}} S(k+1) A \quad (57)$$

$$S(k) = Q + A^{\mathrm{T}} S(k+1)(A - BK(k)) \quad (58)$$

where $S(N) = S_N$. Here, $K(k)$ is the Kalman feedback gain, and $S(k)$ is the solution of the Riccati equation.

2.6. Computational Procedure

From the discussion above, the calculation procedure for estimating the state dynamics and designing the optimal control law is summarized as the computational algorithm. Here, Algorithm 1 is known as the EKF for state-control (EKF4SC) algorithm, and Algorithm 2 is named the UKF for state-control (UKF4SC) algorithm.

Algorithm 1 (EKF4SC Algorithm)

Data: Given f, h, φ, L, A, B, C, G, N, Q, R, $S(N)$, M_0, Q_ω, R_η, $\bar{x}_0$, y.

Step 1: Calculate the state and output error covariance matrices $P(k)$, $M_x(k)$ and $M_y(k)$ from (15), (16), and (17), respectively.

Step 2: Calculate the filter gain $K_f(k)$, feedback gain $K(k)$ and Riccati solution $S(k)$ from (14), (57), and (58), respectively.

Step 3: Compute the state and output estimates $\hat{x}(k)$, $\bar{x}(k)$ and $\bar{y}(k)$ from (11), (12), and (13), respectively.

Step 4: Evaluate the weighted least square error J_{lse} from (10).

Step 5: Compute the feedback control law $u(k)$ from (56).

Step 6: Update the state equation forward in time from (47) to obtain the state solution $\bar{x}(k)$, and solve the costate equation backward in time from (53) to provide the costate solution $p(k)$.

Step 7: Evaluate the cost function J from (43).

Remarks:

(a) In Step 1 and Step 2, the offline calculation is conducted to store the values of matrices.

(b) In Step 3, the state estimation is performed using the EKF approach.

(c) In Step 5, the linear feedback control law is designed.

(d) Step 6 solves a two-point boundary-value problem to give the state and costate solutions.

Algorithm 2 (UKF4SC algorithm)

Data: Given f, h, φ, L, A, B, C, G, N, Q, R, $S(N)$, M_0, Q_ω, R_η, $\bar{x}_0$, and y.

Step 1: Calculate the feedback gain $K(k)$ and the Riccati solution $S(k)$ from (57) and (58), respectively.

Step 2: Prepare the sigma points $\chi(k)$ defined in (32) using the a priori state mean $\bar{x}(k)$ and state error covariance $P(k)$. Next, calculate the predicted state of the transformed sigma points $\chi(k+1)$ from (33), in addition to the mean $\hat{x}^-(k)$ and covariance $P^-(k)$ from (34) and (35), respectively.

Step 3: Compute the output measurement $\Upsilon(k)$ from (36) as well as the mean observation $\hat{y}^-(k)$ and the observation error covariance $P_{yy}(k)$ from (37) and (38), respectively. Subsequently, calculate the cross-correlation matrix $P_{xy}(k)$ from (42) and the Kalman filter gain $K_f(k)$ from (41).

Step 4: Update both the state estimate $\hat{x}(k)$ from (39) and the state error covariance $P(k)$ from (40).

Step 5: Update the state mean $\bar{x}(k)$ forward in time from (47) and solve the costate equation backward in time from (53) so as to provide the costate solution $p(k)$.

Step 6: Compute the feedback control law $u(k)$ from (56).

Step 7: Finally, evaluate the cost function J from (43).

Remarks:

(a) Step 1 is known as the off-line calculation, where the feedback gain $K(k)$ and the Riccati solution $S(k)$ are stored for the later design of the feedback control law.

(b) In Steps 2 and 3, the unscented transform is performed in order to generate a set of sigma points $\chi(k)$ and $\Upsilon(k)$. Meanwhile, in Step 4, the correction step is performed, wherein the output estimate is measured and the state estimate is updated. These steps are known as the state estimation procedure.

(c) During Steps 5 to 7, the two-point boundary-value problem is solved to obtain the solution for the state mean and the costate. Moreover, the feedback control law is designed. These steps are referred to as the system optimization procedure.

3. Results and Discussion

As an illustration, consider the following parameters [5] that are used in the dynamic model (1):

$a = 0.14,\ b = 0.1,\ c = 0.67,\ d = 1$ and $\gamma = 0.5$

with the initial conditions

$x_1(0) = 427.22,\ x_2(0) = 157.1.$

Here, the coefficient matrices are

$$A = \begin{pmatrix} 1.0179 & 0.0064 \\ 0.0000 & 1.0000 \end{pmatrix},\ B = \begin{pmatrix} -46.499 \\ 69.402 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix},\ G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the sampling time is set to $\tau = 0.1$ second and the final time is N = 50. In addition, consider the weighting matrices of the cost function (43), which are $S_N = O_{2\times2}$, $Q = 1\times10^{-4} \times I_{2\times2}$, and $R = 1$. Now, assume that the Gaussian white-noise sequences have the respective covariance $Q_\omega = 100 I_{2\times2}$ and $R_\eta = 100$, while the initial state error covariance is $M_0 = 100 I_{2\times2}$.

The simulation results obtained using the EKF4SC and UKF4SC algorithms are shown in Table 1. The values of the cost function and the performance of the state estimation, as measured on the basis of the sum squares of error (SSE) and the mean squares of error (MSE), are given in the table. The optimal costs derived using these algorithms are almost the same (i.e., 883 units); however, their SSE and MSE values differ. The UKF4SC algorithm provides the smallest SSE and MSE values, which indicates that the output trajectory is statistically closer to the real output observation when compared with the EKF4SC algorithm. As the SSE value is 1.189×10^2, the UKF4SC algorithm is 85.6 percent more efficient than the EKF4SC algorithm. Nonetheless, the performance of the EKF4SC algorithm with an SSE value of 8.282×10^2 can be accepted for the state estimation.

Table 1 Simulation result of economic growth model

Approach	Optimal Cost	SSE	MSE
EKF4SC	883.532125	8.282637×10^2	16.56527
UKF4SC	882.234434	1.189337×10^2	2.378674

The output trajectories determined using the EKF4SC and UKF4SC algorithms are shown in Fig. 1 and 2, respectively. As the real output trajectory is disturbed by the random noise, the estimated output trajectory $\bar{y}(k)$ obtained using the EKF4SC algorithm could track the dynamics of the real output trajectory $y(k)$ at an acceptable level with an MSE value of 16.56. Furthermore, the accuracy of the state estimation in relation to the MSE is increased by around 85.6 percent when the UKF4SC algorithm is employed. It should be noted that the estimated output trajectory of the UKF4SC algorithm is able to precisely meet each data point of the real output trajectory.

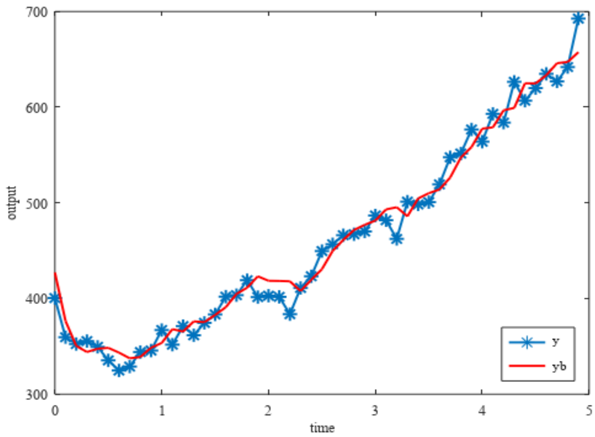

Fig. 1 Output trajectory from EKF4SC algorithm

Fig. 3 and 4 show the state trajectories obtained using the EKF4SC and UKF4SC algorithms, respectively. These state trajectories are quite similar. It can be seen that the final state of the production occurs at $x_1(5)$=707.10 units, which is higher than $x_1(5)$=702.7 units, as mentioned in [5]. Based on the graphical solution, the production is gradually reduced for the first half-year, before it is increased linearly to reach the maximum production. At the same time, the technology stock is increased sharply for the first half-year, while subsequently it increases slowly to reach the maximum value of 438.27 units.

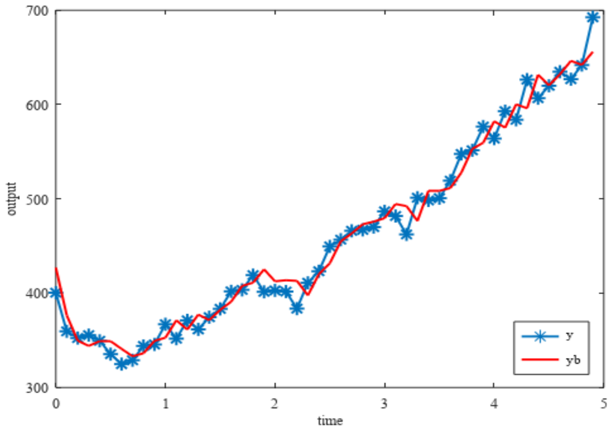

Fig. 2 Output trajectory from the UKF4SC algorithm

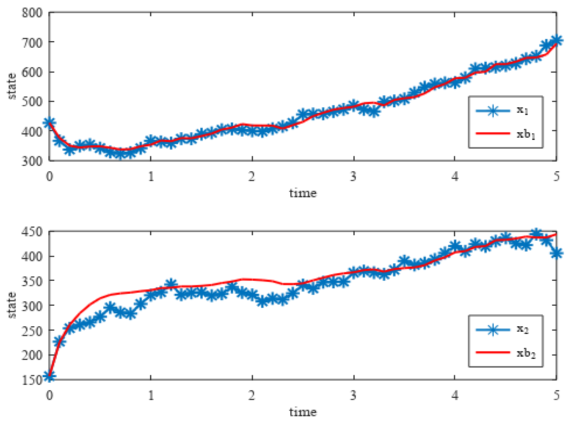

Fig. 3 State trajectory from the EKF4SC algorithm

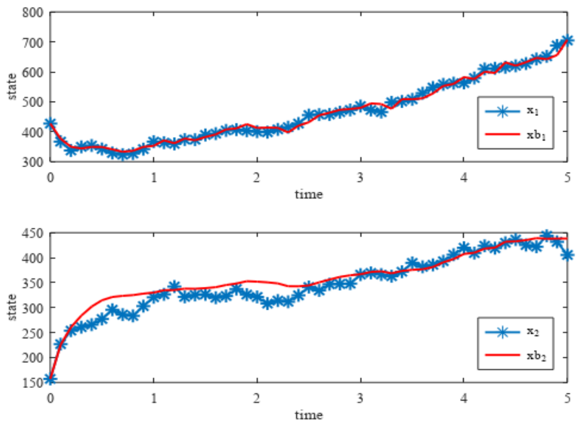

Fig. 4 State trajectory from the UKF4SC algorithm

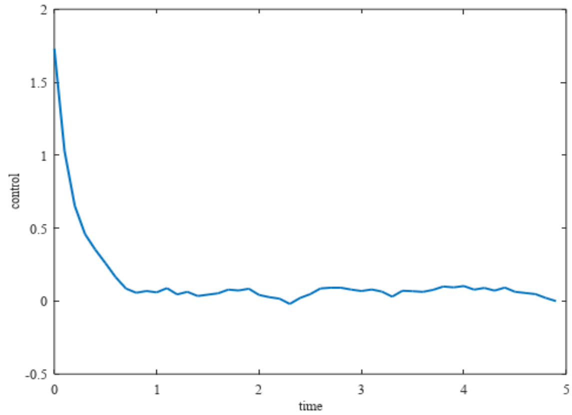

Fig. 5 Control trajectory from the EKF4SC algorithm

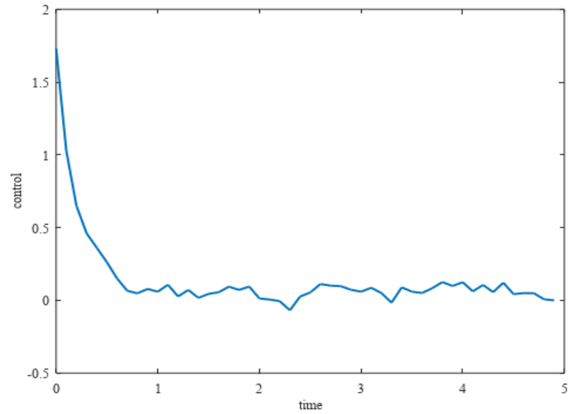

Fig. 6 Control trajectory from the UKF4SC algorithm

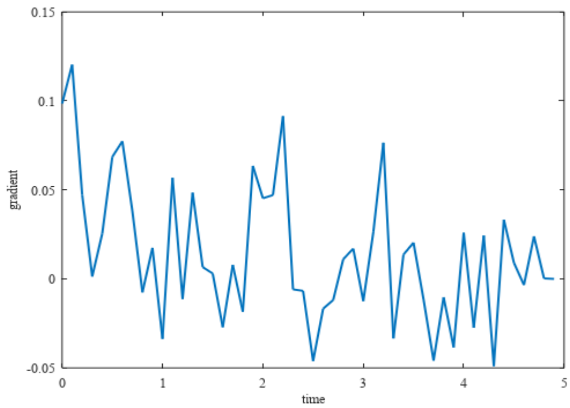

Fig. 7 Stationary condition from the EKF4SC algorithm

The stationary conditions when using the EKF4SC and UKF4SC algorithms are shown in Fig. 7 and 8, respectively. The graphical solution presents the fluctuated curve, which reveals the random noises in the model. Moreover, the values of the curve are between -0.15 and 0.15. From the numerical perspective, these values are small, meaning that they are able to support the satisfaction of the stationary condition. Thus, due to satisfying the stationary condition, the R&D intensity, which is the control input in Problem (P), can be verified in giving the optimal solution for Problem (P).

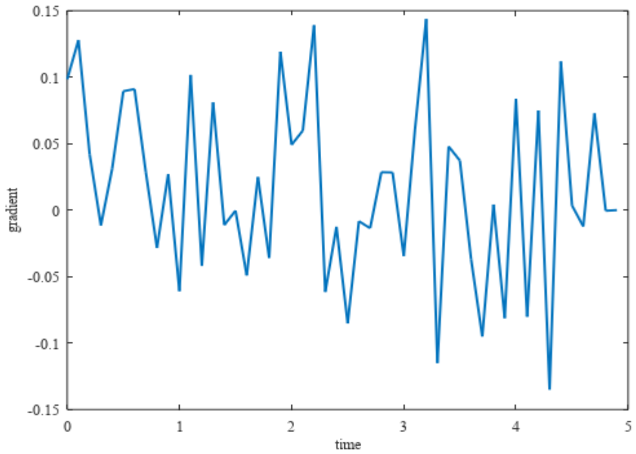

Fig. 8 Stationary condition from UKF4SC algorithm

4. Concluding Remarks

The optimal control of an economic growth model with R&D intensity as the control input was investigated in this paper. More specifically, the stochastic optimal control problem of an economic growth model was formulated by considering the presence of random noises. Kalman filtering approaches, namely the EKF and UKF algorithms, were applied to estimate the state in the problem. Subsequently, the feedback optimal control law was designed based on the state estimate. As an illustration, the values of the model parameters were considered in the simulation. The results revealed that the production and technology stock could be maximized at the final time that the R&D intensity suggested. In conclusion, the efficiency of both algorithms was validated in this study. The significance of this study lies in its ability to provide a stochastic optimal control model for the economic growth problem as well as to suggest an efficient computational approach for solving the stochastic optimal control problem. The algorithms proposed in this study provide an efficient calculation procedure and a satisfactory solution for the stochastic optimal control problem of the economic growth model. However, the MSE value showed that the output estimates still need to be improved in advance so as to approximate the real output as closely as possible. In terms of future studies, some computation methods are suggested so that the accuracy of the solution when it comes to the minimum output error could be enhanced and rendered more accurate.

Acknowledgment

The authors would like to thank the Ministry of Higher Education, Malaysia, for supporting this research under the Fundamental Research Grant Scheme (FRGS), Reference Code FRGS/1/2018/STG06/UTHM/02/5 and partially sponsored by Universiti Tun Hussein Onn Malaysia.

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